Math 151 - Week-In-Review 7

Topics for the week:

3.6 Derivatives of Logarithmic Functions
K.1 Derivatives of Vector Functions
K.2 Slopes and Tangents to Parametric Curves
3.7 Rates of Change in the Natural ~~ans~~ d Social Sciences
3.8 Exponential Growth and Decay

3.6 Derivatives of Logarithmic Functions

1. Compute the derivative of $f(y) = \log_2(y) + 7^y - 7y^{1/7}$ with respect to y.

$$f(y) = \log_2(y) + 7^y - 7y^{1/7}$$

$$\frac{df(y)}{dy} = \frac{1}{y\ln(2)} + 7^y \cdot \ln(7) - y^{-6/7}$$

2. Compute the first and second derivative of $y = \ln\left(4x^2 + 1\right)$.

$$y = \ln(4x^2+1)$$

$$\frac{dy}{dx} = \frac{1}{4x^2+1} \cdot (8x) = \frac{8x}{4x^2+1}$$

$$\frac{d^2y}{dx^2} = \frac{8(4x^2+1) - 8x(8x)}{(4x^2+1)^2}$$

$$= \frac{32x^2 + 8 - 64x^2}{(4x^2+1)^2}$$

$$\frac{d^2y}{dx^2} = \frac{8 - 32x^2}{(4x^2+1)^2}$$

3. For $y = \log(x^2)\tan(x^3)$, find $\frac{dy}{dx}$.

$$y = \log(x^2) \cdot \tan(x^3)$$

$$\frac{dy}{dx} = \frac{1}{x^2 \ln(10)} \cdot 2x \tan(x^3) + \sec^2(x^3)\, 3x^2 \cdot \log(x^2)$$

$$\frac{dy}{dx} = \frac{2\tan(x^3)}{x\ln(10)} + 3x^2 \sec^2(x^3)\log(x^2)$$

4. Compute $f'(0)$ for $f(d) = \log_2\left(e^{-d}\cos(\pi d)\right)$

$f(d) = \log_2\left(e^{-d}\cos(\pi d)\right)$ we could use $\log_2\left(\frac{\cos(\pi d)}{e^d}\right)$

$$f'(d) = \frac{1}{\left(e^{-d}\cos(\pi d)\right)\ln(2)} \cdot \left(-e^{-d}\cos(\pi d) + (-\sin(\pi d)\cdot \pi)e^{-d}\right)$$

$$f'(0) = \frac{-e^{-(0)}\cos(\pi(0)) - \pi e^{-(0)}\sin(\pi(0))}{e^{-(0)}\cos(\pi(0))\ln(2)}$$

$$= \frac{-1\cdot 1 - \pi(1)(0)}{1\cdot 1\cdot \ln(2)}$$

$$f'(0) = \frac{-1}{\ln(2)}$$

5. Determine $\frac{dy}{dx}$ for $y = \sqrt[5]{\frac{x^2+9}{x^2-9}}$

$y = \left(\frac{x^2+9}{x^2-9}\right)^{1/5}$

we could use chain rule here.

$\ln(y) = \ln\left(\left(\frac{x^2+9}{x^2-9}\right)^{1/5}\right)$

$\ln(y) = \frac{1}{5}\ln\left(\frac{x^2+9}{(x-3)(x+3)}\right)$

$\ln(y) = \frac{1}{5}\ln(x^2+9) - \frac{1}{5}\ln(x-3) - \frac{1}{5}\ln(x+3)$

$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{5(x^2+9)}\cdot 2x - \frac{1}{5(x-3)} - \frac{1}{5(x+3)}$

$\frac{dy}{dx} = y\left[\frac{2x}{5(x^2+9)} - \frac{1}{5(x-3)} - \frac{1}{5(x+3)}\right]$

$\frac{dy}{dx} = \sqrt[5]{\frac{x^2+9}{x^2-9}}\left[\frac{2x}{5(x^2+9)} - \frac{1}{5(x-3)} - \frac{1}{5(x+3)}\right]$

6. Determine $\frac{dy}{dx}$ for $y = (\ln(x))^{\sin(x)}$.

$y = (\ln(x))^{\sin x}$

Power rule and Exponential rule may not be directly applied here

$\ln(y) = \ln\left[(\ln(x))^{\sin(x)}\right]$

$\ln(y) = \sin(x)\cdot\ln(\ln(x))$

$\frac{1}{y}\frac{dy}{dx} = \cos(x)\ln(\ln(x)) + \frac{1}{\ln(x)}\cdot\frac{1}{x}\cdot\sin(x)$

Note: $\ln(\ln(x)) \neq \ln^2(x)$

$\frac{dy}{dx} = y\left[\cos(x)\cdot\ln(\ln(x)) + \frac{\sin(x)}{x\ln(x)}\right]$

$\frac{dy}{dx} = (\ln(x))^{\sin x}\cdot\left[\cos(x)\ln(\ln(x)) + \frac{\sin(x)}{x\ln(x)}\right]$

K.1 Derivatives of Vector Functions

7. Determine the derivative of the vector function: $\mathbf{r}(t) = t\cos\left(t^2\right)\mathbf{i} - t^2\sin\left(t\right)\mathbf{j}$.

$\vec{r}(t) = t\cos(t^2)\vec{i} - t^2\sin(t)\vec{j}$

$\vec{r}\,'(t) = \left(1\cos(t^2) + (-\sin(t^2)\cdot 2t)\cdot t\right)\vec{i} + \left((-2t)\sin(t) + \cos(t)(-t^2)\right)\vec{j}$

$\vec{r}\,'(t) = \langle \cos(t^2) - 2t^2\sin(t^2), -2t\sin(t) - t^2\cos(t)\rangle$

8. Compute the unit tangent vector at the point $t = 0$ for $\vec{r}(t) = \left\langle \ln(5t+1), e^{4t}\right\rangle$ with respect to t.

$\dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

$\vec{r}(t) = \langle \ln(5t+1), e^{4t}\rangle$

$\vec{r}\,'(t) = \langle \frac{1}{5t+1}\cdot 5, 4e^{4t}\rangle$

$\vec{r}\,'(0) = \langle \frac{5}{0+1}, 4e^0\rangle = \langle 5, 4\rangle$

$|\vec{r}\,'(0)| = \sqrt{(5)^2+(4)^2} = \sqrt{25+16} = \sqrt{41}$

The unit tangent vector is $\left\langle \frac{5}{\sqrt{41}}, \frac{4}{\sqrt{41}}\right\rangle$

9. Determine the velocity, acceleration, and speed of the particle with a position function of $\mathbf{r}(t) = \left\langle \frac{6t^2}{t-1}, \frac{5}{(t+3)^2} \right\rangle$ ~~with respect to x.~~

$$\vec{v}(t) = \left\langle \frac{12t(t-1) - 1(6t^2)}{(t-1)^2}, 5\cdot(-2)(t+3)^{-3} \right\rangle$$

$$= \left\langle \frac{6t^2-12t}{(t-1)^2}, \frac{-10}{(t+3)^3} \right\rangle$$

$$\vec{a}(t) = \left\langle \frac{(12t-12)(t-1)^2 - 2(t-1)(6t^2-12t)}{(t-1)^4}, -10(-3)(t+3)^{-4} \right\rangle$$

$$= \left\langle \frac{12}{(t-1)^3}, \frac{30}{(t+3)^4} \right\rangle$$

$$\text{Speed} = |\vec{v}(t)| = \sqrt{\left(\frac{6t^2-12t}{(t-1)^2}\right)^2 + \left(\frac{-10}{(t+3)^3}\right)^2}$$

K.2 Slopes and Tangents to Parametric Curves

10. Compute $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dy}{dx}$ for $x = \log(4t+64)$ and $y = t \cdot 10^t$.

$$x = \log(4t+64)$$

$$\frac{dx}{dt} = \frac{4}{\ln(10)\cdot(4t+64)}$$

$$\frac{dx}{dt} = \frac{1}{\ln(10)\cdot(t+16)}$$

$$y = t\cdot 10^t$$

$$\frac{dy}{dt} = 1\cdot 10^t + 10^t \ln(10)\cdot t$$

$$\frac{dy}{dt} = 10^t(1+\ln(10)\,t)$$

$$\frac{dy}{dx} = \frac{10^t\cdot(1+\ln(10)\,t)}{\frac{1}{\ln(10)(t+16)}}$$

$$\frac{dy}{dx} = 10^t(1+\ln(10)\,t)\cdot\ln(10)(t+16)$$

11. Determine an equation of the tangent to the curve $x = \sec(\theta)\tan(\theta)$ and $y = \cos(2\theta)$ at the point corresponding to $\theta = \frac{\pi}{6}$.

$x = \sec(\pi/6)\tan(\pi/6)$

$x = \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}$

$x = \frac{2}{3}$

$y = \cos(2(\pi/6))$

$y = \cos(\pi/3)$

$y = 1/2$

$\left(\frac{2}{3}, 1/2\right)$

$\frac{dx}{d\theta} = \sec(\theta)\tan(\theta)\tan(\theta) + \sec^2(\theta)\sec(\theta)$

$\left.\frac{dx}{d\theta}\right|_{\theta=\pi/6} = \sec(\pi/6)\tan^2(\pi/6) + \sec^3(\pi/6)$

$= \frac{2}{\sqrt{3}} \cdot \frac{1}{3} + \left(\frac{2}{\sqrt{3}}\right)^3$

$= \frac{10}{3\sqrt{3}}$

$\frac{dy}{d\theta} = -\sin(2\theta) \cdot 2$

$\left.\frac{dy}{d\theta}\right|_{\theta=\pi/6} = -2\sin(2 \cdot \pi/6)$

$= -2\sin(\pi/3)$

$= -\sqrt{3}$

$\frac{dy}{dx} = \frac{-\sqrt{3}}{\frac{10}{3\sqrt{3}}} = -\frac{9}{10}$

$$\boxed{y - \frac{1}{2} = -\frac{9}{10}\left(x - \frac{2}{3}\right)}$$

12. Compute the points on the curve where the tangent is horizontal or vertical given $x = 12t - 3t^4$ and $y = t^3 - 9t$.

$x = 12t - 3t^4$

$\frac{dx}{dt} = 12 - 12t^3$

Vertical tangent $\Rightarrow \frac{dx}{dt} = 0$

$0 = 12 - 12t^3$

$12t^3 = 12$

$t^3 = 1$

$\boxed{t = 1}$

$(12-3, 1-9)$

$= (9, -8)$

$y = t^3 - 9t$

$\frac{dy}{dt} = 3t^2 - 9$

Horizontal tangent $\Rightarrow \frac{dy}{dt} = 0$

$0 = 3t^2 - 9$

$9 = 3t^2$

$3 = t^2$

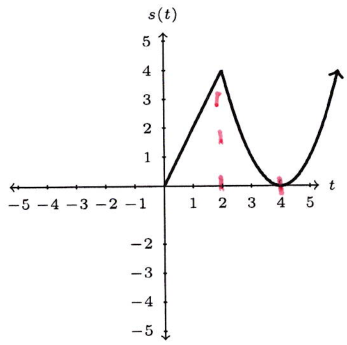
$\pm\sqrt{3} = t$

$(12\sqrt{3} - 27, -6\sqrt{3})$ $t = \sqrt{3}$

$(-12\sqrt{3} - 27, 6\sqrt{3})$ $t = -\sqrt{3}$

3.7 Rates of Change in the Natural and Social Sciences

13. Given the graph of a position function of a particle is shown below, where t is measured in seconds.



(a) When is the velocity of the particle positive?

when position is increasing

$(0,2)$ and $(4,\infty)$

(b) When is the particle not moving?

when position changes directions or is undefined

$t=2, 4$

(c) When is the particle moving backwards?

when the velocity is negative or when the position is decreasing

$(2,4)$

14. A particle moves according to the function $s(t) = \frac{1}{3}t^3 - 2t^2 + 3t + 4$, where t is in seconds and $s(t)$ is in meters.

(a) Compute $v(t)$ and $a(t)$.

$v(t) = t^2 - 4t + 3$

$a(t) = 2t - 4$

(b) When is the particle at rest?

$v(t) = 0$

$t^2 - 4t + 3 = 0$

$(t-3)(t-1) = 0$

$t = 3 \quad t = 1$

at $t = 1$ sec and $t = 3$ sec

(c) How far did the particle travel in the first 4 seconds?

distance over each interval

$[0,1]$ $\quad |s(1) - s(0)| = \left|\left(\frac{1}{3} - 2 + 3 + 4\right) - (0 - 0 + 0 + 4)\right| = \frac{4}{3}$ meters

$[1,3]$ $\quad |s(3) - s(1)| = \left|\left(\frac{1}{3}(27) - 2(3)^2 + 9 + 4\right) - \left(\frac{1}{3} - 2 + 3 + 4\right)\right| = \frac{4}{3}$ meter

$[3,4]$ $\quad |s(4) - s(3)| = \left|\left(\frac{1}{3}(4)^3 - 2(4)^2 + 12 + 4\right) - \left(\frac{1}{3}(27) - 2(3)^2 + 9 + 4\right)\right| = \frac{4}{3}$ meter

Total distance $= \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4$ meters

Exponential Growth and Decay

3.8 ~~Rates of Change in the Natural ans Social Sciences~~

15. A chemical has a half-life of 18 days. A sample is obtained and 5 days later there remains 50 grams of the chemical.

(a) Write a formula that will give the amount of the chemical that remains t days after the sample is obtained.

$$A(t) = A_0 e^{rt}$$
$$\frac{1}{2}A_0 = A_0 e^{r(18)}$$
$$\frac{1}{2} = e^{18r}$$
$$\ln(\tfrac{1}{2}) = \ln(18r)$$
$$\ln(1/2) = 18r$$
$$\frac{1}{18}\ln(1/2) = r \quad \text{or} \quad -\frac{\ln(2)}{18} = r$$
$$\ln(2^{-1/18}) = r$$

$$A(t) = A_0 \cdot e^{\ln(2^{-1/18})t}$$
$$= A_0 e^{\ln(2^{-\frac{1}{18}t})}$$
$$A(t) = A_0 (2)^{-\frac{1}{18}t}$$
$$50 = A_0 (2)^{-\frac{1}{18}(5)}$$
$$\frac{50}{2^{-5/18}} = A_0 = 50 \cdot 2^{5/18}$$
$$A(t) = \frac{50}{2^{-5/18}} \cdot (2)^{-\frac{1}{18}t}$$
$$A(t) = 50\left(2^{-\frac{1}{18}t + \frac{5}{18}}\right)$$

(b) What was the initial amount of the sample of this chemical?

$$A_0 = \frac{50}{2^{-5/18}} \approx 60.6163 \text{ grams}$$
$$\text{or } 50 \cdot 2^{5/18}$$

(c) Determine the rate of decay of the chemical after 4 days.

$$A(t) = \frac{50}{2^{-5/18}} \cdot \left(2^{-1/18t}\right)$$
$$A'(t) = \frac{50}{2^{-5/18}} \cdot \left(2^{-1/18t}\right) \cdot \ln(2) \cdot \left(-\frac{1}{18}\right)$$
$$A'(4) = \frac{-50\ln(2)}{2^{-5/18}(18)} \cdot (2)^{-1/18(4)} = \frac{-50\ln(2) \cdot 2^{1/18}}{18} \approx -2.001 \text{ g/d}$$