Math 308: Week-in-Review 3

1. Determine (without solving the problem) an interval in which the solution of the following initial value problem is certain to exist.

$y' + p(x)y = g(x) \Rightarrow$ solution exists where $p(x), g(x)$ are cts

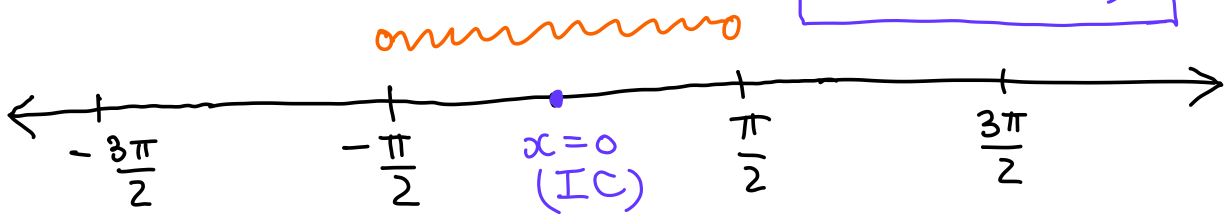
(a)

$$y' + (\sec x)y = x^2, \quad y(0) = 5$$

$p(x) = \sec(x) = \frac{1}{\cos(x)} \rightarrow$ discontinuous at odd multiples of $\frac{\pi}{2}$

$g(x) = x^2 \rightarrow$ continuous everywhere

$\boxed{\text{Domain: } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$



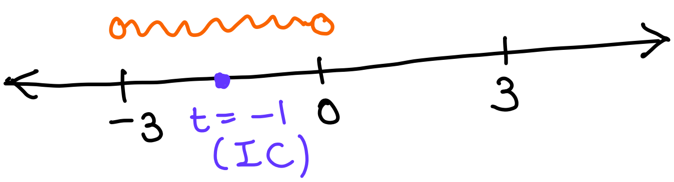
(b)

$$y' + \frac{t}{t^2-9}y = \frac{1}{t}, \quad y(-1) = 2$$

$p(t) = \frac{t}{t^2-9}$ discontinuous at $t = \pm 3$

$g(t) = \frac{1}{t}$ discontinuous at $t = 0$

$\boxed{\text{Domain: } (-3, 0)}$

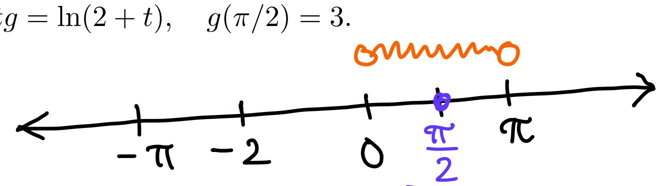


(c)

defined on $t > -2$

$$\sin(t)g' + 2tg = \ln(2+t), \quad g(\pi/2) = 3.$$

$g' + \frac{2t}{\sin(t)}g = \frac{\ln(2+t)}{\sin(t)}$



$p(t) = \frac{2t}{\sin(t)}$ discontinuous at multiples of π

$0, \pm\pi, \pm 2\pi, \ldots$

$\boxed{\text{Domain: } (0, \pi)}$

2. (a) Consider the differential equation

$$y' = (2t + y)^{\frac{2}{3}}.$$

If the initial condition is $y(0) = 1$, does the IVP have a unique solution? What if the initial condition is $y(1) = -2$?

(a) The Existence & Uniqueness Theorem (EUT)

IF (a) $f(t,y)$ & $\frac{\partial f}{\partial y}(t,y)$ are continuous at (t_0, y_0)

then there $\boxed{\text{exists}}$ a $\boxed{\text{unique solution}}$ to $y'(t) = f(t,y)$, $y(t_0) = y_0$.

$y(0) = 1$: * $f(t,y) = (2t+y)^{2/3} \Rightarrow f(0,1) = (2\cdot 0 + 1)^{2/3} = 1$ ✓ cts

* $\frac{\partial f}{\partial y}(t,y) = \frac{2}{3}(2t+y)^{-1/3}$

$= \frac{2}{3(2t+y)^{1/3}} \Rightarrow \frac{\partial f}{\partial y}(0,1) = \frac{2}{3(2\cdot 0+1)^{1/3}} = \frac{2}{3}$ ✓ cts

EUT $\Rightarrow$ $y' = (2t+y)^{2/3}$, $y(0)=1$ has a unique solution

$y(1) = -2$: * $f(1,-2) = (2\cdot 1 + (-2))^{2/3} = 0^{2/3}$ ✓ cts

* $\frac{\partial f}{\partial y}(1,-2) = \frac{2}{3(2\cdot 1 + (-2))^{1/3}} = \frac{2}{3\cdot 0}$ ✗ undefined

$y' = (2t+y)^{2/3}$, $y(1) = -2$ is not guaranteed to have a unique solution passing through $(1,-2)$ in the t-y plane.

separable

(b) Consider the initial value problem $y' = \sin(2t)y^{\frac{1}{3}}$, $y(0) = 0$. One solution is $y(t) = 0$. Find two other solutions to the initial value problem. Why does the Existence and Uniqueness Theorem not apply to this case?

$y(t) = 0$ is one solution. Find others.

$y \neq 0$:

$$\int y^{-1/3}\,dy = \int \sin(2t)\,dt$$

$$\frac{3}{2} y^{2/3} = -\frac{1}{2}\cos(2t) + C \qquad y(0)=0 \Rightarrow C = \frac{1}{2}$$

$$\frac{3}{2} y^{2/3} = -\frac{1}{2}\cos(2t) + \frac{1}{2} = \underbrace{\frac{1}{2}\left(1-\cos(2t)\right)}_{\sin^2(t)}$$

$$y^{2/3} = \frac{2}{3}\sin^2(t)$$

$$y^2 = \left(\frac{2}{3}\sin^2(t)\right)^3$$

$$\boxed{y = \pm\sqrt{\left(\frac{2}{3}\sin^2(t)\right)^3} = \pm\sqrt{\frac{8}{27}}\sin^3(t)}$$

$$f(t,y) = \sin(2t)\,y^{1/3} \Rightarrow f(0,0) = \sin(2\cdot 0)\,0^{1/3} = 0 \checkmark \text{ continuous}$$

$$\frac{\partial f}{\partial y} = \frac{\sin(2t)}{3y^{2/3}} \Rightarrow \frac{\partial f}{\partial y}(0,0) = \frac{\sin(2\cdot 0)}{3\cdot 0^{2/3}} \times \text{ undefined}$$

Therefore EUT does not apply to this case

3. Solve the following initial value problems and determine how the interval in which the solution exists depends on y_0.

Equilibrium solns : $y=0$. If $y_0=0$, the solution $y(t)=0$ exists for all t values

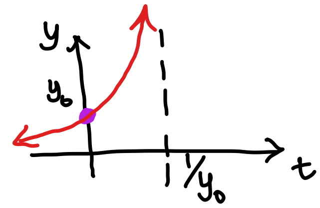
(a) $y' = y^2, \quad y(0) = y_0$

$y_0 \neq 0$: $\int \frac{1}{y^2} dy = \int 1 \cdot dt$

$\Rightarrow -y^{-1} = t + C \Rightarrow -\frac{1}{y} = t + C \Rightarrow -\frac{1}{y_0} = C \Rightarrow \frac{1}{y} = \frac{1}{y_0} - t \Rightarrow \boxed{y = \frac{1}{1 - ty_0}}$

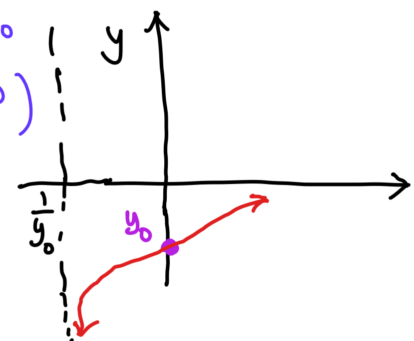
Case 1: $y_0 > 0$: vertical asymptote at $t = \frac{1}{y_0} > 0$

Domain of solution : $(-\infty, \frac{1}{y_0})$



Case 2: $y_0 < 0$: vertical asymptote at $t = \frac{1}{y_0} < 0$

Domain of solution : $(\frac{1}{y_0}, \infty)$



(b) $y' = -\frac{4t}{y}, \quad y(0) = y_0$

$* \; y \neq 0 \Rightarrow y(0) = 0$ has no solns

$y \neq 0$: $\int y \, dy = -4 \int t \, dt$

$\frac{y^2}{2} = -\frac{4}{2} t^2 + C_1 \Rightarrow y^2 = -4t^2 + \overbrace{2C_1}^{C} = -4t^2 + C, \; y(0) = y_0 \Rightarrow y_0^2 = C$

$y^2 + 4t^2 = y_0^2 \Rightarrow y^2 = y_0^2 - 4t^2 \Rightarrow y = \pm\sqrt{y_0^2 - 4t^2}$

$\Downarrow$

$y_0^2 - 4t^2 > 0 \Leftarrow$ strict! $y \neq 0$

$t^2 < \frac{y_0^2}{4}$

$$\boxed{|t| < \frac{|y_0|}{2}}$$

4. Determine if the following equations are autonomous or not.

↳ the independent variable does not appear explicitly ⇒ the form of the equation does not change with the independent variable

(a) $f''(x) - 3f(x)f'(x) + 4 = 0$

$f'' - 3ff' + 4 = 0$

autonomous

(b) $\dfrac{q''(x)}{x^2+1} - q(x)^{3/2} = 4\cos(x)$

$\dfrac{q''}{x^2+1} - q^{3/2} = 4\cos(x)$

non-autonomous

(c) $y'' + y' + y = 0$

autonomous

(d) $\dfrac{g''}{g^2} + g = \sqrt{g}$

autonomous

(e) $\dfrac{d^2y}{dx^2} + 3(x^2-1)y - x = 5\sin(2x)$

$y'' + 3(x^2-1)y - x = 5\sin(2x)$

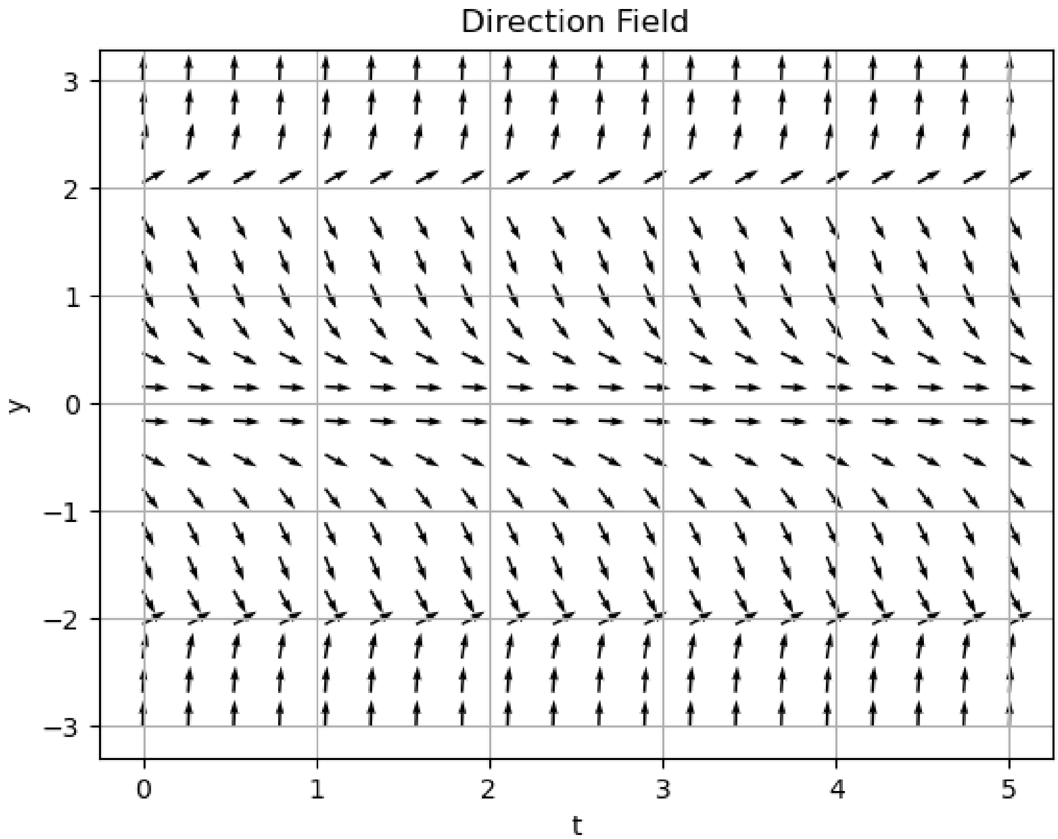
non-autonomous

(f) $\sin(u^3) + \dfrac{d^3u}{dx^3} = 0$

$\sin(u^3) + u''' = 0$

autonomous

5. Given the following slopefield, determine the equilibrium solutions and their stability. Also, draw the phaseline diagram.

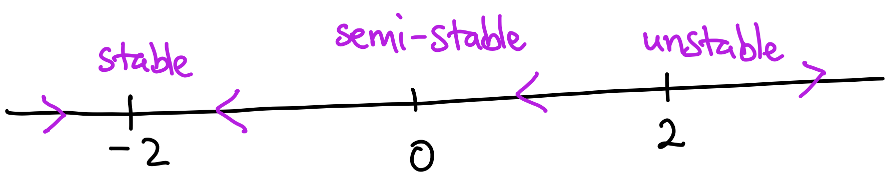


unstable

semistable

stable

(a) Equilibrium solutions: → zero slope $y=-2$, $y=0$, $y=2$



6. Given the differential equation

$$y' = (1+y)(y-2)^2 = f(y)$$

(a) Find the equilibrium solutions.

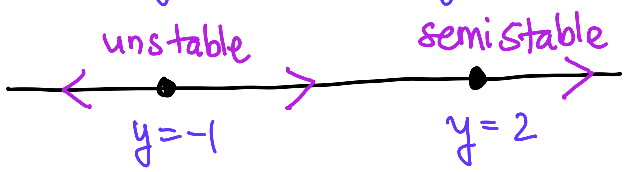
(b) Graph the phase line. Classify each equilibrium solution as either stable, unstable, or semistable

(c) Graph some solutions

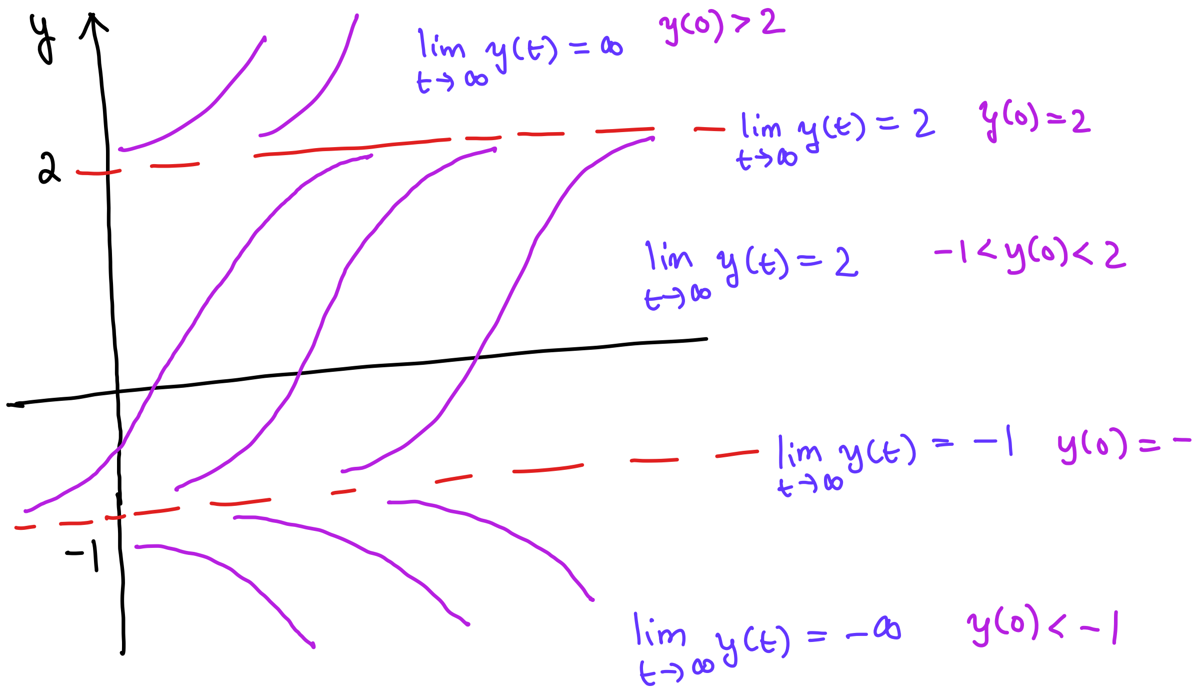
(d) If $y(t)$ is the solution of the equation satisfying the initial condition $y(0) = y_0$ for some $y_0 \in (-\infty, \infty)$, find the limit of $y(t)$ as $t \to \infty$

(a) $f(y) = (1+y)(y-2)^2 = 0 \Leftrightarrow y = -1,\ y = 2$

(b)



(c)



7. Suppose that the population of rabbits obeys the logistic equation

$$\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{1000}\right).$$

* logistic growth
* carrying capacity = 1000

Initially there are $P(0) = 100$ rabbits.

(a) Solve the differential equation to find the population $P(t)$ as a function of time.

(b) Find the time it takes for the population to reach 90% of the carrying capacity.

(a) Equilibrium solns: $P=0$, $P=1000$. But $P(0)=100 \leftarrow$ not an eq. soln

(b) Separation of variables: $\frac{1}{P\left(1-\frac{P}{1000}\right)}dP = 0.1\,dt$

$$\int \frac{1}{P\left(1-\frac{P}{1000}\right)}dP = \int 0.1\,dt = 0.1t + C = \frac{t}{10} + C$$

* partial fractions * $\frac{1}{P\left(1-\frac{P}{1000}\right)} = \frac{A}{P} + \frac{B}{1-\frac{P}{1000}} \Rightarrow 1 = A\left(1-\frac{P}{1000}\right) + BP$

$$\boxed{\begin{array}{l} P=0: \Rightarrow A = 1 \\ P=1000: \Rightarrow B = \frac{1}{1000} \end{array}}$$

$$\int \frac{1}{P\left(1-\frac{P}{1000}\right)}dP = \int \frac{1}{P}dP + \int \frac{1}{1000-P}dP = \ln|P| - \ln|1000-P| = \ln\left|\frac{P}{1000-P}\right| = \frac{t}{10} + c$$

$$\left|\frac{P}{1000-P}\right| = e^{c}\cdot e^{t/10} \Rightarrow \frac{P}{1000-P} = \pm e^{c}\cdot e^{t/10} = Ce^{t/10}.$$ Find C. $P(0)=100$

$$\frac{100}{1000-100} = C \Rightarrow C = \frac{100}{900} = \frac{1}{9} \Rightarrow P = \frac{1}{9}e^{t/10}(1000-P) \Rightarrow P\left(1+\frac{1}{9}e^{t/10}\right) = \frac{1000}{9}e^{t/10}$$

$$P = \frac{\frac{1000}{9}e^{t/10}}{1+\frac{1}{9}e^{t/10}}\cdot\frac{\left(9e^{-t/10}\right)}{\left(9e^{-t/10}\right)} = \frac{1000}{9e^{-t/10}+1} \Rightarrow \boxed{P = \frac{1000}{9e^{-t/10}+1}}$$

(c) Solve: $\overbrace{900}^{90\%} = \frac{1000}{9e^{-t/10}+1} \Rightarrow \frac{9}{10} = \frac{1}{9e^{-t/10}+1} \Rightarrow 9e^{-t/10}+1 = \frac{10}{9}$

$$\Rightarrow 9e^{-t/10} = \frac{1}{9} \Rightarrow e^{-t/10} = \frac{1}{81} \Rightarrow -\frac{t}{10} = \ln\left(\frac{1}{81}\right) \Rightarrow \frac{t}{10} = \ln(81) = \ln(3^4)$$

$$\boxed{t = 10\ln(3^4) = 40\ln(3) \approx 43.94}$$

8. (a) Show that the equation is exact and find the general solution

$$\overset{M}{(2x\sin y + y^3e^x)} + \overset{N}{(x^2\cos y + 3y^2e^x)}\frac{dy}{dx} = 0.$$

1. Check for exactness: $\underbrace{\frac{\partial M}{\partial y} = 2x\cos y + 3x^2e^x, \quad \frac{\partial N}{\partial x}}_{\text{exact!}} = 2x\cos y + 3y^2e^x$

2. Therefore there exists $F(x,y)$ such that $\frac{\partial F}{\partial x} = M = 2x\sin y + y^3e^x$ and $\frac{\partial F}{\partial y} = x^2\cos y + 3y^2e^x$.

3. Find $F(x,y)$: $F(x,y) = \int M\,dx = \int (2x\sin y + y^3e^x)\,dx = x^2\sin y + y^3e^x + h(y)$

$\frac{\partial F}{\partial y} = \underbrace{x^2\cos y + 3y^2e^x + h'(y)}_{N} \Rightarrow h'(y) = 0 \Rightarrow h(y) = C$

$F(x,y) = x^2\sin y + y^3e^x + c$

4. Solution: $\boxed{x^2\sin y + y^3e^x = C}$

(b) Show that the equation is exact and find the solution to the initial value problem

$$\overset{M}{(2t\cos y + 3t^2y)} + \overset{N}{(t^3 - t^2\sin y - y)}\frac{dy}{dt} = 0, \quad y(0) = 2.$$

1. Check for exactness: $\frac{\partial M}{\partial y} = -2t\sin y + \underbrace{3t^2, \frac{\partial N}{\partial t}}_{\text{exact!}} = 3t^2 - 2t\sin y$

2. Set $\frac{\partial F}{\partial t} = M = 2t\cos y + 3t^2y$ and $\frac{\partial F}{\partial y} = N = t^3 - t^2\sin y - y$

3. Find $F(x,y)$: $F(t,y) = \int (t^3 - t^2\sin y - y)\,dy = t^3y + t^2\cos y - \frac{y^2}{2} + h(t)$

$\frac{\partial F}{\partial t} = \underbrace{3t^2y + 2t\cos y + h'(t)}_{M} \Rightarrow h'(t) = 0 \Rightarrow h(t) = C$

$F(t,y) = t^3y + t^2\cos y - \frac{y^2}{2} + C$

4. Solution: $\left.t^3y + t^2\cos y - \frac{y^2}{2} = C\right\}$ $y(0) = 2 \Leftrightarrow t=0, y=2 \Rightarrow C = -2$

$\boxed{t^3y + t^2\cos y - \frac{y^2}{2} = -2}$