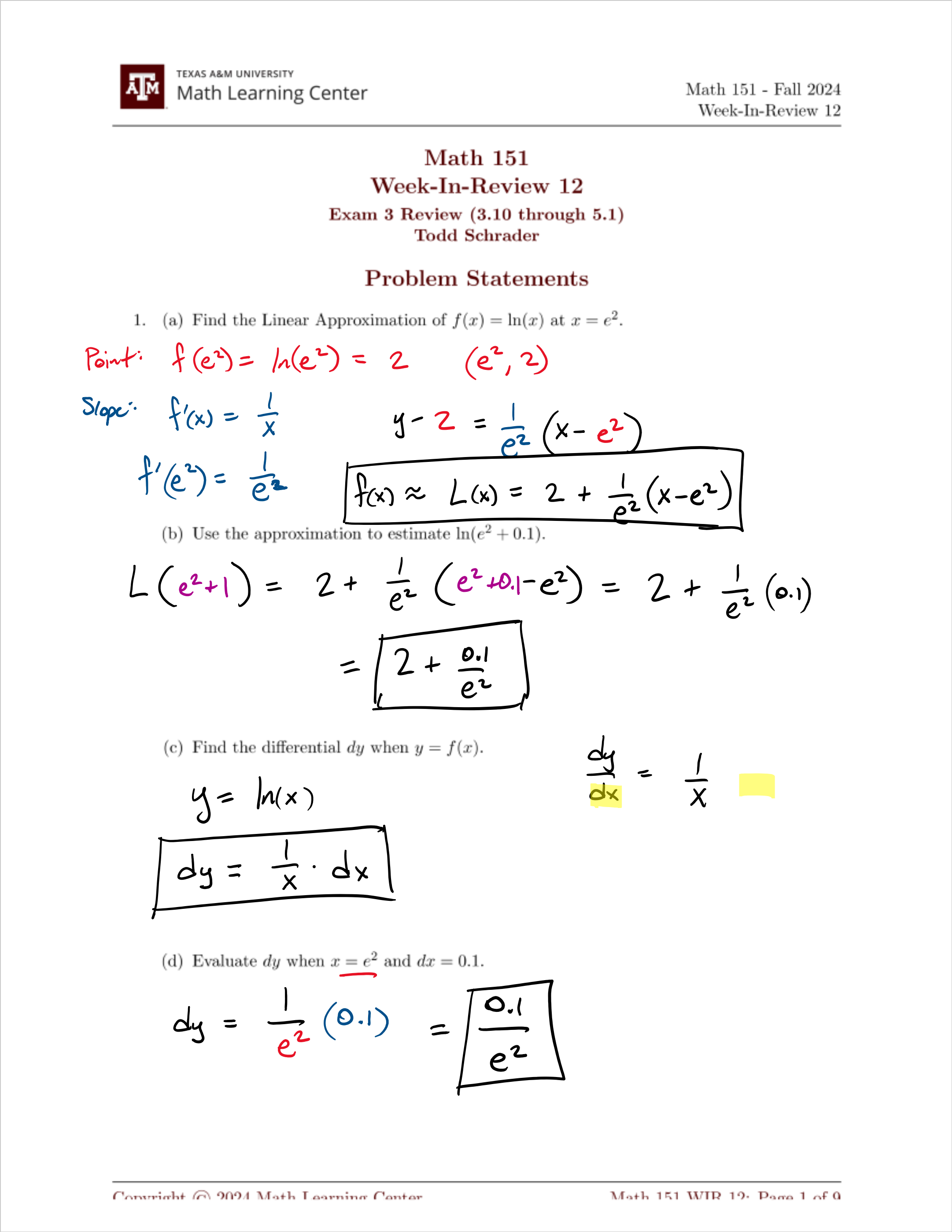
Week 12

Monday, November 11, 2024 7:23 PM



Math 151
Week-In-Review 12
Exam 3 Review (3.10 through 5.1)
Todd Schrader

Problem Statements

1. (a) Find the Linear Approximation of $f(x) = \ln(x)$ at $x = e^2$.

Point: $f(e^2) = \ln(e^2) = 2$ $(e^2, 2)$

Slope: $f'(x) = \frac{1}{x}$

$f'(e^2) = \frac{1}{e^2}$

$y - 2 = \frac{1}{e^2}(x - e^2)$

$$\boxed{f(x) \approx L(x) = 2 + \frac{1}{e^2}(x - e^2)}$$

(b) Use the approximation to estimate $\ln(e^2 + 0.1)$.

$$L(e^2+1) = 2 + \frac{1}{e^2}(e^2 + 0.1 - e^2) = 2 + \frac{1}{e^2}(0.1)$$

$$= \boxed{2 + \frac{0.1}{e^2}}$$

(c) Find the differential dy when $y = f(x)$.

$\frac{dy}{dx} = \frac{1}{x}$

$y = \ln(x)$

$$\boxed{dy = \frac{1}{x} \cdot dx}$$

(d) Evaluate dy when $x = e^2$ and $dx = 0.1$.

$$dy = \frac{1}{e^2}(0.1) = \boxed{\frac{0.1}{e^2}}$$

2. (a) Determine the absolute maximum (and minimum) of $h(x) = \dfrac{5x^2+8x+5}{x}$ on the interval [0.2, 2].

$$h'(x) = \frac{x(10x+8) - (5x^2+8x+5)(1)}{x^2} = \frac{10x^2+8x-5x^2-8x-5}{x^2}$$

$$h'(x) = \frac{5x^2-5}{x^2} = \frac{5(x^2-1)}{x^2} = \frac{5(x-1)(x+1)}{x^2}$$

Absolute Minimum is 18
it occurs @ $x=1$

$h'(x)$ DNE : $x^2=0$ $x=0$

x	h(x)
1	18
0.2	$\frac{5(\frac{1}{5})^2 + \frac{8}{5} + 5}{(1/5)} = 34$
2	$\frac{5(2)^2+8(2)+5}{2} = \frac{41}{2} = 20.5$

Absolute Maximum is 34
it occurs @ $x = 0.2$

$h'(x) = 0$ $\frac{5(x-1)(x+1)}{x^2} = 0$

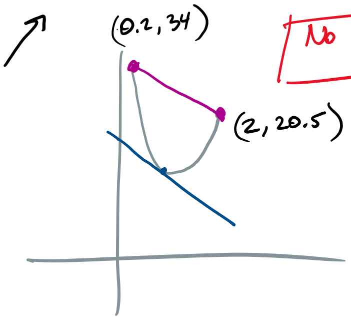
$5(x-1)(x+1) = 0$

$x-1=0$ $x+1=0$

$x=1$ $x=-1$

(b) How does the answer change on the interval (0.2, 2)?

	$(\frac{1}{2})$	(2)
	0.2 (−)(+)	(+)(+)
$h'(x)$:	−	+
$h(x)$:	↘	↗



Absolute Minimum occurs @ $x=1$
Min is 18.

No Absolute Maximum

$$m = \frac{20.5-34}{2-0.2} = \frac{-13.5}{1.8}$$

$$= \frac{-13.5}{9/5} = \frac{-27}{2} \cdot \frac{5}{9} = \frac{-15}{2} = \underbrace{\frac{5x^2-5}{x^2}}^{h'(x)}$$

$-15x^2 = 10x^2 - 10$

$-25x^2 = -10$

$x^2 = \frac{2}{5}$

$x = \pm\sqrt{2/5}$

(c) Determine the value(s) of x that satisfy the conclusion of the Mean Value Theorem on the interval [0.2, 2].

(0.2, 2) $x = \sqrt{2/5} \approx 0.6$

$\ln(\boxed{2x+5})$ $\quad 2x+5>0$

$$\frac{0}{\#}=0 \qquad \frac{\#}{0}=\infty \qquad \frac{0}{0}=? \qquad \frac{\infty}{\#}=\infty \qquad \frac{\#}{0}=\infty$$

3. Consider $f(x)=\dfrac{\ln x}{x}$. Find the following:

(a) Domain, Asymptotes, and Intercepts

$x \neq 0$

$\ln(x) \rightarrow x>0$

$\boxed{D:\ (0,\infty)}$

Vertical Asymptotes

$$\lim_{x\to 0^+}\frac{\ln(x)}{x}=-\infty \qquad \frac{\ln(0)}{0}=\frac{-\infty}{0^+}$$

$\boxed{V.A.\quad x=0}$

Horizontal Asymptotes:

$$\cancel{\lim_{x\to-\infty} f(x)}\ \text{DNE}$$

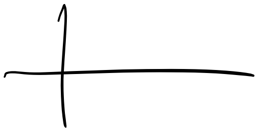
$$\lim_{x\to\infty}\frac{\ln(x)}{x} \qquad \frac{\ln(\infty)}{\infty}=\frac{\infty}{\infty}\ \checkmark$$

$$\overset{L'H}{=}\lim_{x\to\infty}\frac{1/x}{1}=\frac{1}{\infty}=0$$

$\boxed{H.A.\quad y=0}$

x-int: $y=0$

$$0=\frac{\ln x}{x}$$



$$e^{\ln(x)}=e^{0} \qquad x=e^{0} \qquad x=1$$

$\boxed{(1,0)\quad x\text{-int}}$

y-int. $(x=0)$ Not in Domain

$\boxed{\text{No y-int}}$

$f(x)=\dfrac{\ln(x)}{x}$

(b) Intervals of Increase/Decrease, Locations of Local Extrema

$$f'(x)=\frac{x\cdot\frac{1}{x}-\ln(x)\cdot 1}{x^2}=\frac{1-\ln(x)}{x^2}$$

$f'(x)$ DNE: $x=0$

$f'(x)=0 \qquad \dfrac{1-\ln(x)}{x^2}=0$

$1-\ln(x)=0$

$e^{\ln(x)}=e^{1}$

$x=e$

$\boxed{\begin{array}{l} f(x)\ \text{Inc.}:\ (0,e) \\ f(x)\ \text{Dec}:\ (e,\infty)\end{array}}$

(1) $\qquad (e^{-5}) \qquad (e^{10})$

$0 \quad e$

$f'(x)$: $1-\ln(e^{-5})$, $1-(-5)$, $+$ $\quad$ $1-\ln(e^{10})$, $1-10$, $-$

$f(x)$: ↗ $\quad$ ↘

$\boxed{\begin{array}{l}\text{Local Max @ } x=e \\ \text{No Local min.}\end{array}}$

$$f'(x)=\frac{1-\ln(x)}{x^2}$$

(c) Intervals of Concavity, Locations of Inflection Points

$$f''(x)=\frac{x^2\cdot\left(-\frac{1}{x}\right)-(1-\ln x)\cdot 2x}{x^4}$$

$$=\frac{-x-2x+2x\ln(x)}{x^4}$$

$$=\frac{-3x+2x\ln(x)}{x^4}$$

$f''(x)$ DNE: $x=0$

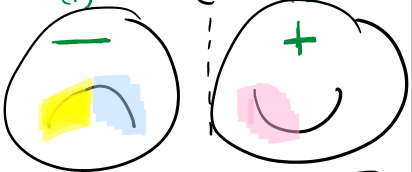
$(1) \qquad (e^{10})$

$0 \quad e^{3/2}$

$\frac{(-)}{(+)} \qquad \frac{(+)}{(+)}$

$f''(x)$: $-$ $\quad$ $+$

$f(x)$:



$$= \frac{-3x + 2x\ln(x)}{x^4}$$

$$= \frac{x\,(-3 + 2\ln x)}{x^4}$$

$$f''(x) = \frac{-3 + 2\ln(x)}{x^3} = 0$$

$f''(x)$ DNE:

$X = 0$

$f''(x) = 0$

$-3 + 2\ln(x) = 0$

$2\ln(x) = 3$

$\ln(x) = \frac{3}{2}$

$X = e^{3/2}$

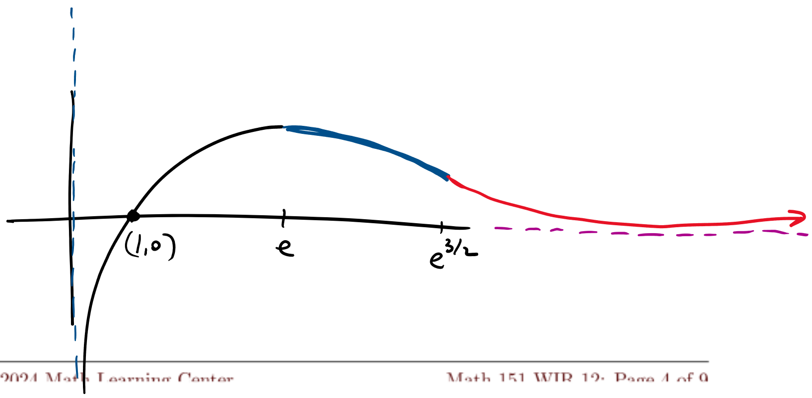
Concave Up: $(e^{3/2}, \infty)$

Concave Down: $(0, e^{3/2})$

Inflection Point

@ $X = e^{3/2}$

(d) Sketch a Graph



4. Evaluate the following limits.

(a) $\lim\limits_{x\to -1}\dfrac{x^3+4x^2+5x+2}{\frac{1}{3}x^3-\frac{1}{2}x^2-2x-\frac{7}{6}}$ $\overset{L'H}{=}\lim\limits_{x\to-1}\dfrac{3x^2+8x+5}{x^2-x-2}$

$\dfrac{-1+4-5+2}{-\frac{1}{3}-\frac{1}{2}+2-\frac{7}{6}}=\dfrac{0}{0}$ ✓ $\dfrac{3-8+5}{1+1-2}=\dfrac{0}{0}$ ✓

$$\overset{L'H}{=}\lim_{x\to-1}\frac{6x+8}{2x-1}=\frac{6(-1)+8}{2(-1)-1}=\boxed{\frac{2}{-3}}$$

(b) $\lim\limits_{t\to\infty} te^{-t^2+1}$ $=\lim\limits_{t\to\infty}\dfrac{t}{e^{-(-t^2+1)}}=\lim\limits_{t\to\infty}\dfrac{t}{e^{t^2-1}}$ $\dfrac{\infty}{e^{\infty}}=\dfrac{\infty}{\infty}$ ✓

$\infty\cdot e^{-\infty+1}$

$\infty\cdot 0$

$$\overset{L'H}{=}\lim_{t\to\infty}\frac{1}{e^{t^2-1}(2t)}=\frac{1}{e^{\infty}\cdot 2(\infty)}=\frac{1}{\infty\cdot(\infty)}=\frac{1}{\infty}=\boxed{0}$$

(c) $\lim\limits_{r\to e^+}[\ln(r)]^{1/(r-e)}$ $=L$ $\ln(L)=\ln\left(\lim\limits_{r\to e^+}[\ln(r)]^{1/(r-e)}\right)$

$[\ln(e)]^{1/(e^+-e)}$

1^{∞}

$$\ln(L)=\lim_{r\to e^+}\frac{1}{r-e}\ln[(\ln(r))]$$

$\ln(L)=\dfrac{1}{e}$

$$=\lim_{r\to e^+}\frac{\ln(\ln r)}{r-e}\overset{L'H}{=}\lim_{r\to e^+}\frac{\frac{1}{\ln(r)}\cdot\frac{1}{r}}{1}$$

$\boxed{L=e^{1/e}}$

$\ln(\ln e)=\ln(1)=0$

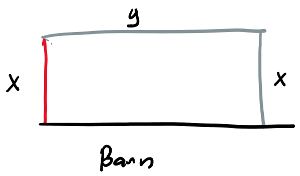
$e-e=0$ $\dfrac{0}{0}$ ✓

$$=\frac{1}{\ln(e)}\cdot\frac{1}{e}=\frac{1}{1}\cdot\frac{1}{e}=\frac{1}{e}$$

5. A farmer wants to fence in a rectangular plot of land adjacent to the north wall of his barn. No fencing is needed along the barn, and the fencing along the west side of the plot is shared with a neighbor who will split the cost of that portion of the fence. If the fencing costs $20 per linear foot to install and the farmer is not willing to spend more than $5000, find the dimensions for the plot that would enclose the most area. Verify your answer produces a maximum.

Maximize Area: $A = x \cdot y$

Split w/neighbor



Cost: $20x + 20y + 10x$

$30x + 20y = 5000$

$20y = 5000 - 30x$

$y = \frac{5000}{20} - \frac{3}{2}x$

$y = 250 - \frac{3}{2}x$

$A = x\left(250 - \frac{3}{2}x\right)$

$A = 250x - \frac{3}{2}x^2$

$A' = 250 - 3x$

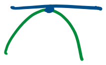
A' DNE: N/A

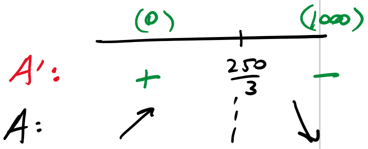
$A' = 0 \quad 250 - 3x = 0$

$3x = 250$

$x = \frac{250}{3}$

$A'' = -3$





$x = \frac{250}{3}$

$y = 250 - \frac{3}{2}\left(\frac{250}{3}\right)$

$y = 125$

6. If the farmer instead wanted to enclose 8000 square feet of land, what dimensions will minimize the cost of the fence? Verify your answer produces a minimum.

7. Find the most general antiderivative of the following functions.

(a) $f(x) = x^8 - 4e^x + \sin(x)$

$$\boxed{F(x) = \frac{1}{9}x^9 - 4e^x + -\cos(x) + C}$$

* Typo.

(b) $g(x) = \dfrac{x^3 - 2x^{3/2}\sec^2(x) + 3\overset{x^{1/2}}{x^{5/2}}}{x^{3/2}} = x^{3/2} - 2\sec^2(x) + 3\overset{\frac{1}{x}}{x^{-1}}$

$$\boxed{G(x) = \frac{2}{5}x^{5/2} - 2\tan(x) + 3\ln|x| + C}$$

$5(1+t^2)^{-1}$

$5 \cdot \dfrac{1}{1+t^2}$

$\dfrac{d}{dt}\left[\ln(1+t^2)\right]$

$\dfrac{1}{1+t^2} \cdot 2t$

8. If $f'(t) = \cos(t) + \dfrac{5}{1+t^2}$ and $f(1) = \sin(1)$, find $f(t)$.

$f(t) = \sin(t) + 5\arctan(t) + C$

$f(1) = \sin(1) + 5\arctan(1) + C = \sin(1)$

$5\arctan(1) + C = 0$

$\tan(\theta) = 1$

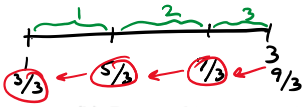
$5\left(\frac{\pi}{4}\right) + C = 0$

$C = -\frac{5\pi}{4}$

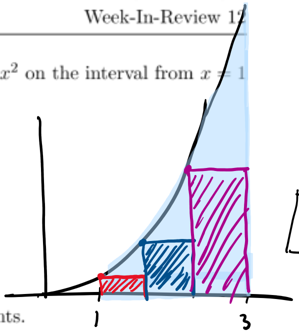
$$\boxed{f(t) = \sin(t) + 5\arctan(t) - \frac{5\pi}{4}}$$

(a) Estimate the area under the curve of the function $f(x) = x^2$ on the interval from $x = 1$ to $x = 3$ with using three rectangles and left endpoints.

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{3} = \frac{2}{3}$$



$x_2^* = \frac{5}{3}$



$$A \approx \frac{2}{3} f(1) + \frac{2}{3} f\left(\frac{5}{3}\right) + \frac{2}{3} f\left(\frac{7}{3}\right)$$

$$\boxed{A \approx \frac{2}{3}\left(1^2 + \left(\frac{5}{3}\right)^2 + \left(\frac{7}{3}\right)^2\right)}$$

(b) Repeat this process with four rectangles and right endpoints.

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$$

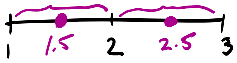
$$A \approx \frac{1}{2} f(1.5) + \frac{1}{2} f(2) + \frac{1}{2} f(2.5) + \frac{1}{2} f(3)$$

$$\boxed{A \approx \frac{1}{2}\left[(1.5)^2 + 2^2 + (2.5)^2 + 3^2\right]}$$

$x_4^* = 3$

(c) Repeat this process with two rectangles and midpoints.

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{2} = \frac{2}{2} = 1$$



$$A \approx 1 \cdot f(1.5) + 1 \cdot f(2.5)$$

$$\boxed{A \approx 1 \cdot \left[(1.5)^2 + (2.5)^2\right]}$$

(d) How does the answer change if you estimate the area of $g(x) = x^2 + 5$?

(e) Set up a limit that would represent the exact area under the curve of $f(x) = x^2$ on the interval from $x = 1$ to $x = 3$.

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$$

x_i^* = Sample Points $= 1 + i \cdot \frac{2}{n} = 1 + \frac{2i}{n}$

$= a + i \cdot \Delta x$ ← Right Endpoints

$$A \approx \sum_{i=1}^{n} \frac{2}{n} f(x_i^*)$$

$$\boxed{A = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2}{n} \cdot \left(1 + \frac{2i}{n}\right)^2}$$

9. An object is moving with a velocity of 100 m/s when it undergoes a constant acceleration $a(t) = -8$ m/s^2. How far does the object travel before coming to a stop?

$v(t) = 0$

$a(t) = -8$ $\qquad$ $t = 0 \Rightarrow v(0) = 100$

$v(t) = -8t + C_1$ $\qquad$ $100 = 0 + C_1$ $\quad$ $C_1 = 100$

$v(t) = -8t + 100 \longrightarrow 0 = -8t + 100$ $\qquad$ $8t = 100$

$t = 12.5$

$s(t) = -8 \cdot \frac{1}{2} t^2 + 100t + C_2$

$s(t) = -4t^2 + 100t + C_2$

$S(12.5) = -4(12.5)^2 + 100(12.5) + C_2$

$- \; S(0) = C_2$

$$\boxed{-4(12.5)^2 + 100(12.5)}$$

Distance

10. Determine a region whose area is equal to the given limit. Do not evaluate the limit.

(a) $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{3}{n} \sqrt{1 + \frac{3i}{n}}$

Add $\quad$ Width Δx $\quad$ $f(x_i^*)$

$\Delta x = \frac{3}{n} = \frac{b-a}{n}$ $\qquad$ $b - a = 3$

$b - 1 = 3$ $\quad$ $b = 4$

$x_i^* = a + i \cdot \Delta x = 1 + i \cdot \frac{3}{n}$

$a = 1$ $\qquad$ $b = 4$

$f(x) = \sqrt{x}$

$$\boxed{f(x) = \sqrt{x} \quad \text{on} \quad (1, 4)}$$

(b) $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{i\pi^2}{16n^2} \tan\left(\frac{i\pi}{4n}\right) = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{\infty} \underbrace{\frac{\pi}{4n}}_{\Delta x} \cdot \underbrace{\frac{\pi}{4n} \cdot i \; \tan\left(\frac{\pi}{4n} \cdot i\right)}_{x_i^* \cdot \tan(x_i^*)}$

$\Delta x = \frac{\pi/4}{n} = \frac{b-a}{n}$

$b - a = \frac{\pi}{4}$

$x_i^* = a + i\Delta x = 0 + i\,\frac{\pi/4}{n}$

$a = 0$ $\qquad$ $b = \pi/4$

$$\boxed{f(x) = x \cdot \tan(x) \quad \text{on} \quad \left(0, \frac{\pi}{4}\right)}$$