Math 140: Week-in-Review 6 (4.2, 4.3, 4.4 & Chapters 3 & 4 review)

1. An Easter egg is chosen from a box of containing 4 yellow, 2 green, 1 blue, 5 red, and 3 orange eggs and its color is noted.

Y G B R O

$S = \{Y, G, B, R, O\}$ → sample space

(a) Write the probability distribution for this experiment.

X = color chosen (outcome)

total number of eggs $= 4+2+1+5+3 = 15$

X	Y	G	B	R	O
$P(x)$	$\frac{4}{15}$	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{5}{15}$	$\frac{3}{15}$

* it is encouraged NOT to simplify/reduce these fractions! → easier arithmetic

(b) Does this experiment have a uniform sample space? Explain why or why not.

* uniform if the probability of each outcome is the same *

NO, the sample space is NOT uniform because the probabilities of the outcomes are not the same

(c) What is the probability that a yellow or a blue egg is chosen?

$Y \cup B$

$$P(Y \cup B) = P(X=Y) + P(X=B)$$
$$= \frac{4}{15} + \frac{1}{15} = \frac{5}{15}$$

(d) What is the probability that an orange egg is not chosen?

O^c

$$P(O^c) = 1 - P(X=O)$$
$$= 1 - \frac{3}{15}$$
$$= \frac{12}{15}$$

2. A pair of fair **five-sided** dice are rolled noting the uppermost numbers.

(a) Write an appropriate sample space for this experiment.

$S = \{(1,1), (1,2), (1,3), (1,4), (1,5),$
$(2,1), (2,2), (2,3), (2,4), (2,5),$
$(3,1), (3,2), (3,3), (3,4), (3,5),$
$(4,1), (4,2), (4,3), (4,4), (4,5),$
$(5,1), (5,2), (5,3), (5,4), (5,5)\}$

all possible outcomes

number of outcomes $= 5 \times 5 = 25$

(b) Is this a uniform sample space? Explain why or why not.

Yes, this is a uniform sample space because the probability of obtaining each outcome is the same, i.e. $\frac{1}{25}$

* sum is 4 or less

(c) Write the event, E, that a sum of less than 5 is rolled.

$E = \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1)\}$

(d) Determine $P(E)$.

$$P(E) = \frac{\#\text{ of outcomes in } E}{\#\text{ of outcomes in } S} = \frac{6}{25}$$

(e) Determine the probability that a sum of 3 and at least one 4 is rolled.

$\{(1,2), (2,1)\} \cap \{\text{at least one } 4\}$

↳ sum is at least 5

P(sum of 3 and at least one 4) = 0 (impossible event)

∪ combine

(f) Determine the probability that a product of 12 or a 5 is rolled on at least one of the dice.

{(3,4),(4,3)} {(1,5),(2,5),(3,5),(4,5),(5,5),(5,1),(5,2),(5,3),(5,4),(5,5)}

product of 12 — at least one 5 is rolled

$P(\text{product of 12 OR at least one 5}) = \frac{11}{25}$

∪ combine

(g) Determine the probability that a sum of 7 is rolled or a double is rolled.

{(2,5),(3,4),(4,3),(5,2)} {(1,1),(2,2),(3,3),(4,4),(5,5)}

$P(\text{sum of 7 OR a double}) = \frac{9}{25}$

(h) Determine the probability that a 2 is not rolled on either die. * use complement rule since it is easier to count the complement here *

$P(\text{2 is not rolled on either dice}) = 1 - P(\text{2 is rolled on at least one dice})$

$= 1 - \frac{9}{25} = \frac{16}{25}$

(i) Draw a probability distribution for the outcome, X, representing the sum of the numbers rolled.

$S = \{2,3,4,5,6,7,8,9,10\}$

X	2	3	4	5	6	7	8	9	10
P(x)	$\frac{1}{25}$	$\frac{2}{25}$	$\frac{3}{25}$	$\frac{4}{25}$	$\frac{5}{25}$	$\frac{4}{25}$	$\frac{3}{25}$	$\frac{2}{25}$	$\frac{1}{25}$

* check sum $= \frac{25}{25} = 1$ ✓

(j) Determine the expected value for the sum, X.

$E(X) = (2)\left(\frac{1}{25}\right) + (3)\left(\frac{2}{25}\right) + (4)\left(\frac{3}{25}\right) + (5)\left(\frac{4}{25}\right) + (6)\left(\frac{5}{25}\right) + (7)\left(\frac{4}{25}\right)$

$+ (8)\left(\frac{3}{25}\right) + (9)\left(\frac{2}{25}\right) + (10)\left(\frac{1}{25}\right)$

$= \frac{1}{25}(2+6+12+20+30+28+24+18+10) = \frac{150}{25} = 6$

3. The students attending an Investment Club meeting were surveyed concerning the courses they were taking in the Fall. The findings were gathered in the table below

	Math (M)	Economics (E)	Accounting (A)	Other (O)	Totals
Freshman (F)	17	10	27	5	59
Sophomore (H)	12	7	13	9	41
Totals	29	17	40	14	100

For each question below, re-write the question being asked using probability notation and then determine the numerical answer.

Determine the probability that a randomly surveyed student at the Investment Club meeting

(a) Is a sophomore?

$$P(H) = \frac{41}{100}$$

(b) Is taking a Math class?

$$P(M) = \frac{29}{100}$$

$\cap$ common to both

(c) Is a sophomore and taking a Math class?

$$P(H \cap M) = \frac{12}{100}$$

$\cup$ combine

(d) Is a sophomore or taking a Math class?

$$P(H \cup M) = \frac{41 + 17}{100} = \frac{58}{100}$$

(e) Is not taking an Accounting class?

$$P(A^c) = 1 - P(A) = 1 - \frac{40}{100} = \frac{60}{100}$$

$\cup$ combine

(f) Is a freshman or is not taking an Accounting class?

$$P(F \cup A^c) = \frac{100 - 13}{100} = \frac{87}{100}$$

4. Let $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ be the sample space for an experiment with the following probability distribution.

Outcome	s_1	s_2	s_3	s_4	s_5	s_6
Probability	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{5}{16}$	$\frac{1}{16}$	$\frac{3}{16}$

* sum $= \frac{16}{16}$ ✓

(a) Determine if the probability distribution is uniform. Explain why or why not.

The probability distribution is not uniform because the probabilities of the outcomes are not the same

(b) Assume that $A = \{s_1, s_4, s_6\}$, $B = \{s_2, s_4, s_5\}$ and $C = \{s_1, s_3, s_6\}$. Determine the following probabilities.

(i) $P(A)$

$$P(A) = P(\{s_1\}) + P(\{s_4\}) + P(\{s_6\})$$
$$= \frac{2}{16} + \frac{5}{16} + \frac{3}{16} = \frac{10}{16}$$

(ii) $P(B)$

$$P(B) = P(\{s_2\}) + P(\{s_4\}) + P(\{s_5\})$$
$$= \frac{3}{16} + \frac{5}{16} + \frac{1}{16} = \frac{9}{16}$$

(iii) $P(A \cup B)$ $A \cup B = \{s_1, s_2, s_4, s_5, s_6\} \Rightarrow (A \cup B)^c = \{s_3\}$

$$P(A \cup B) = 1 - P(A \cup B)^c = 1 - P(\{s_3\}) = 1 - \frac{2}{16}$$
$$= \frac{14}{16}$$

(iv) $P(B \cap C)$ $B \cap C = \{\ \} = \phi$

$$P(B \cap C) = P(\{\ \}) = 0$$

Outcome	s_1	s_2	s_3	s_4	s_5	s_6
Probability	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{5}{16}$	$\frac{1}{16}$	$\frac{3}{16}$

(v) $P(B \cup C)$

$B \cup C = \{s_1, s_2, s_3, s_4, s_5, s_6\} \to$ contains <u>all</u> outcomes

$P(B \cup C) = 1$

(vi) $P(A^C)$

$\frac{10}{16}$ from part (a)

$$P(A^c) = 1 - P(A)$$
$$= 1 - \frac{10}{16}$$
$$= \frac{6}{16}$$

(vii) $P(B^C \cup A)$

$B = \{s_2, s_4, s_5\}$, $B^c = \{s_1, s_3, s_6\}$

$A = \{s_1, s_4, s_6\}$

$B^c \cup A = \{s_1, s_3, s_4, s_6\}$

$$P(B^c \cup A) = P(\{s_1\}) + P(\{s_3\}) + P(\{s_4\}) + P(\{s_6\})$$
$$= \frac{2}{16} + \frac{2}{16} + \frac{5}{16} + \frac{3}{16}$$
$$= \frac{12}{16}$$

5. Suppose that $P(E) = 0.45$, $P(F) = 0.55$, and $P(E \cup F) = 0.8$. Calculate the following.

(a) $P(E \cap F)$

* union rule: $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

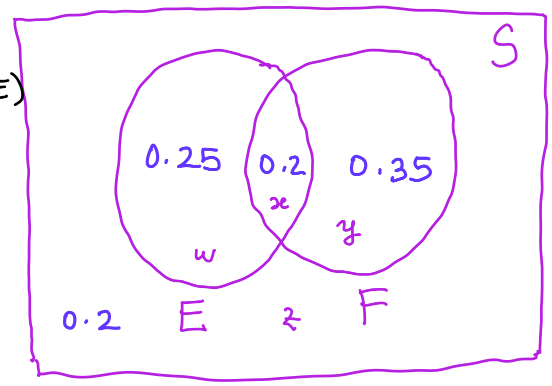
$$0.8 = 0.45 + 0.55 - P(E \cap F)$$

$$\Rightarrow 0.8 = 1 - \underbrace{P(E \cap F)}_{0.2}$$

$P(E \cap F) = \underline{0.2}$

(b) $P(E^C)$

OR $P(E^c) = 1 - P(E)$
$= 1 - 0.45$
$= \underline{0.55}$



$E^c = \{y, z\}$

$P(E^c) = 0.35 + 0.2$
$= \underline{0.55}$

(c) $P(E^C \cap F^C)$

$E^c = \{y, z\}$, $F^c = \{w, z\}$

$E^c \cap F^c = \{z\}$

$P(E^c \cap F^c) = \underline{0.2}$

OR Using DeMorgan's Laws

$E^c \cap F^c = (E \cup F)^c$

$P(E^c \cap F^c) = P(E \cup F)^c$
$= 1 - P(E \cup F)$
$= 1 - 0.8$
$= \underline{0.2}$

(d) $P(E^C \cup F)$

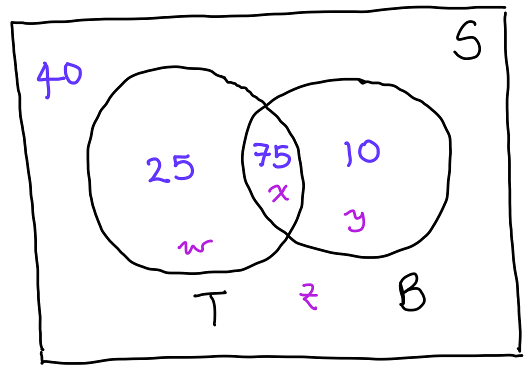
$E^c = \{y, z\}$, $F = \{x, y\}$

$E^c \cup F = \{x, y, z\}$

$P(E^c \cup F) = 0.2 + 0.35 + 0.2$
$= 0.75$

$T^c \cap B^c$

6. 150 students were surveyed about the sports they play. 100 students play tennis (T), 85 play basketball (B), and 40 don't play either sport. Determine the probability that a randomly selected student



T = play tennis = 100

B = play basketball = 85

$T^c \cap B^c$ = don't play either sport = 40

Recall DeMorgan's Laws

$T^c \cap B^c = (T \cup B)^c = \{z\} = 40$

(a) Plays both tennis and basketball. ($T \cap B$)

$$P(T \cup B) = P(T) + P(B) - P(T \cap B)$$

* re-arrange to get *

$$P(T \cap B) = P(T) + P(B) - P(T \cup B)$$

$$= \frac{100}{150} + \frac{85}{150} - [1 - \underbrace{P(T \cup B)^c}_{\frac{40}{150}}]$$

$$= \frac{100}{150} + \frac{85}{150} - \frac{110}{150} = \frac{75}{150}$$

(b) Plays only basketball

$$P(\text{only basketball}) = P(y)$$

$$= \frac{10}{150}$$

(c) Does not play tennis?

$$P(T^c) = 1 - P(T)$$

$$= 1 - \frac{100}{150}$$

$$= \frac{50}{150}$$

OR

$T^c = \{y, z\}$

$$P(T^c) = \frac{10 + 40}{150}$$

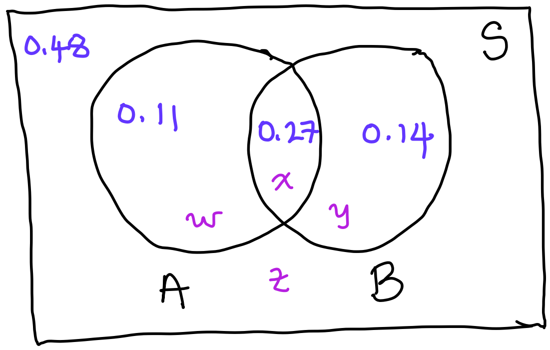
$$= \frac{50}{150} \checkmark$$

7. Assume $P(A) = 0.38$ and $P(B^C) = 0.59$

(a) If $P(A \cap B) = 0.27$, determine

(i) $P(A \cup B)$

$$P(A \cup B) = 0.11 + 0.27 + 0.14$$
$$= \underline{0.52}$$



$P(B) = 1 - P(B^c) = 1 - 0.59 = 0.41$

(ii) $P(A^C \cap B)$

$$P(A^c \cap B) = P(y)$$
$$= \underline{0.14}$$

$A^c = \{y, z\}$, $B = \{x, y\}$

$A^c \cap B = \{y\}$

$A \cap B = \{\}$

(b) If A and B are mutually exclusive, determine

(i) $A \cap B = \{\}$

(ii) $P(A \cap B) = P(\{\}) = 0$

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ 0

$$= P(A) + P(B)$$
$$= 0.38 + 0.41$$
$$= \underline{0.79}$$

$P(B) = 1 - P(B^c)$
$= 1 - 0.59$
$= 0.41$

(vi) $P(A^C \cup B^C)$

$$P(A^c \cup B^c) = P(S) = \underline{1}$$

De Morgan's Laws

$A^c \cup B^c = (A \cap B)^c$
$= \{\}^c$ mutually exclusive
$= S$

8. Consider the probability distribution below

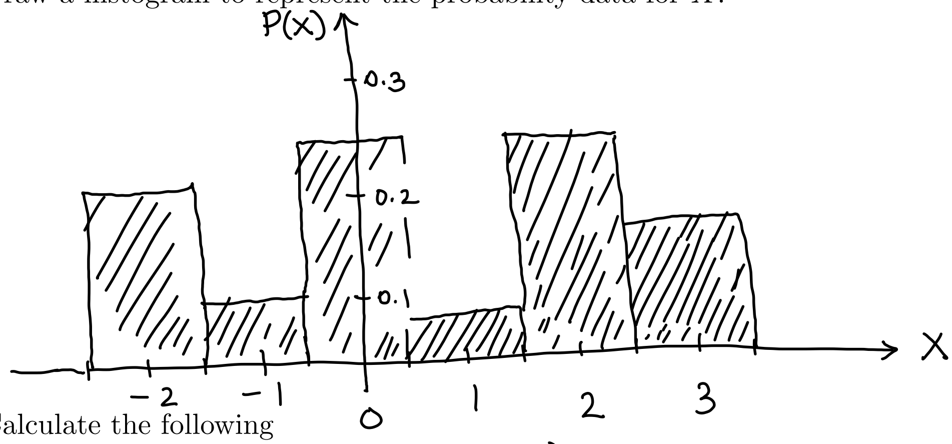
X	-2	-1	0	1	2	3
$P(X)$	0.2	0.1	0.25	0.05	0.25	0.15

* sum = 1

(a) Fill in the missing probability in the probability distribution above.

$1-(0.2+0.1+0.25+0.05+0.15) = 1-0.75 = 0.25$

(b) Draw a histogram to represent the probability data for X.



(c) Calculate the following

(i) $P(X<0) = P(X=-2)+P(X=-1)$

$= 0.2+0.1$

$= 0.3$

(ii) $P(-2<X<3) = P(X=-1)+P(X=0)+P(X=1)+P(X=2)$

$= 0.1+0.25+0.05+0.25$

$= 0.65$

(iii) $P(X=1.5) = 0$ (since $X \neq 1.5$)

(d) Determine the expected value of X.

$E(X) = (-2)(0.2)+(-1)(0.1)+(0)(0.25)+(2)(0.25)+(3)(0.15)$

$= -0.4-0.1+0+0.5+0.45$

$= 0.45$

9. An artwork is insured for \$15,000 in the case that it is stolen and \$7,500 in the case that it is damaged over the next year. There is a 0.5% chance that the artwork will be stolen in the next year, and a 2% chance it will be damaged in the next year. The annual premium charged by the insurance company is \$$p$ → premium

X = company profit = premium − pay off

(a) Write down the probability distribution for the insurance company's profit on this policy

	stolen	damaged	nothing happens
X	$p - 15{,}000$	$p - 7500$	$p - 0$
$P(x)$	0.005	0.02	$1-(0.005+0.02)$ 0.975

* sum = 1

expected profit = \$0

(b) Determine the minimum premium charged by the insurance company to break-even.

$$E(X) = (p-15000)(0.005) + (p-7500)(0.02) + p(0.975) = 0$$

$$\Rightarrow p - 75 - 150 = 0$$

$$\Rightarrow p - 225 = 0$$

$$\boxed{p = \$225}$$ → minimum premium charged

10. You pay $2 to play a game where you roll a fair **four sided die** and toss a **fair coin**. If the coin comes up heads you win twice the amount shown on the die in dollars. If the coin comes up tails and you roll an odd number, you win the amount shown on the die in dollars. Otherwise you win nothing.

win − $2

(a) Draw a probability distribution table to describe your **net** winnings.

(T,2),(T,4)

Outcome	(H,1)	(H,2)	(H,3)	(H,4)	(T,1)	(T,3)	Otherwise
X = net winnings	2 − 2 = 0 $0	4 − 2 = 2 $2	6 − 2 = 4 $4	8 − 2 = 6 $6	1 − 2 = −1 −$1	3 − 2 = 1 $1	0 − 2 = −2 −$2
P(X)	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{2}{8}$

* experiment outcomes

$S = \{(H,1),(H,2),(H,3),(H,4),(T,1),(T,2),(T,3),(T,4)\}$

* uniform since each outcome has probability $\frac{1}{8}$

(b) Determine your expected net winnings.

$$E(X) = (0)\left(\tfrac{1}{8}\right) + (2)\left(\tfrac{1}{8}\right) + (4)\left(\tfrac{1}{8}\right) + (6)\left(\tfrac{1}{8}\right) - (1)\left(\tfrac{1}{8}\right) + (1)\left(\tfrac{1}{8}\right) - (2)\left(\tfrac{2}{8}\right)$$

$$= \frac{8}{8} = \boxed{\$1}$$

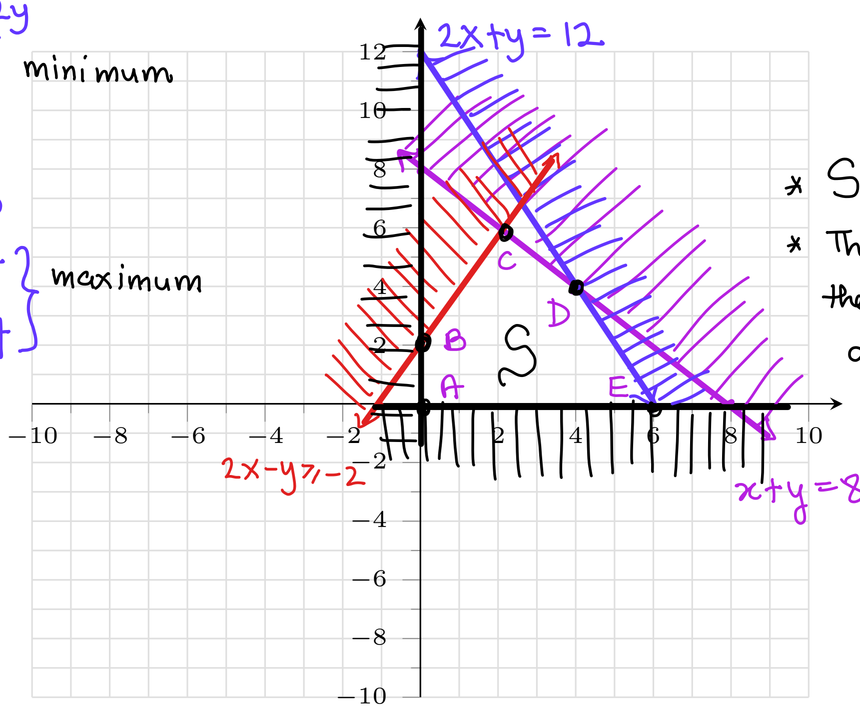
(c) Is the game fair or unfair? Explain.

A game is fair if the expected NET winnings, $E(X) = \$0$. Otherwise it is unfair.

11. The following inequalities are constraints in a linear programming problem. Graph the inequalities and determine if the region is bounded or unbounded. Determine all corner points.

$$\begin{aligned} x + y &\leq 8 \\ 2x + y &\leq 12 \\ 2x - y &\geq -2 \\ x \geq 0,\ \ y &\geq 0. \end{aligned}$$

Corner pnt	$z = 4x + 2y$
A: (0,0)	$z = 0$ * minimum
B: (0,2)	$z = 4$
C: (2,6)	$z = 20$
D: (4,4)	$z = 24$ maximum
E: (6,0)	$z = 24$ maximum



* S is bounded

* The FTLP implies the objective function attains a minimum and a maximum at a corner pnt of S

C: $2x - y = -2,\ x + y = 8$

$$\left[\begin{array}{cc|c} 2 & -1 & -2 \\ 1 & 1 & 8 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 6 \end{array}\right]$$

$(x,y) = (2,6)$

D: $2x + y = 12,\ x + y = 8$

$$\left[\begin{array}{cc|c} 2 & 1 & 12 \\ 1 & 1 & 8 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 4 \end{array}\right]$$

$(x,y) = (4,4)$

(a) At what point(s) is the objective function $z = 4x + 2y$ maximized on this region, and what is the maximum value? (If there is no maximum value, explain why not.)

The max value is $z = 24$ and occurs at all points joining the corner points D (4,4) and E (6,0).

(b) At what point(s) is the objective function $z = 4x + 2y$ minimized on this region, and what is the minimum value? (If there is no minimum value, explain why not.)

The minimum value is $z = 0$ at corner pnt A (0,0)

12. A company makes souvenirs of type A and B that they sell to tourists. Each A is sold for \$20 and each B is sold for \$17. It takes 3 hours and \$12 to make each A and 1 hour and \$15 to make each B. If the company has a total of 90 hours and \$600 available for making the souvenirs, and the company can not make more than 20 type A souvenirs because of limited demand, how many of each souvenir should the company make in order to maximize their revenue? What is the maximum revenue? Is anything leftover at the optimal production level?

(a) Formulate and then solve the given linear programming using the Simplex Method.

Maximize $R = 20x + 17y$

Subject to: $3x + y \le 90$ (time)

$12x + 15y \le 600$ (capital)

$x \le 20$ (max of A)

$x \ge 0,\ y \ge 0$ (non-negative)

$x = $ # of A
$y = $ # of B
$R = $ revenue generated

$3x + y + s_1 = 90$
$12x + 15y + s_2 = 600$
$x + s_3 = 20$
$-20x - 17y + P = 0$

x	y	s_1	s_2	s_3	P	const	
3	1	1	0	0	0	90	$\frac{90}{3} = 30$
12	15	0	1	0	0	600	$\frac{600}{12} = 50$
1	0	0	0	1	0	20	$\frac{20}{1} = 20$
−20	−17	0	0	0	1	0	not optimal

pivot row → (third row)

pivot column ↑ (x column)

x	y	s_1	s_2	s_3	P	const	
0	1	1	0	−3	0	30	$30/1 = 31$
0	15	0	1	−12	0	360	$\frac{360}{15} = 24$
1	0	0	0	1	0	20	$\frac{20}{0}$ X
0	−17	0	0	20	1	400	not optimal

pr → (second row)

pc ↑ (y column)

x	y	s_1	s_2	s_3	P	const
0	0	1	$-\frac{1}{15}$	$-\frac{11}{5}$	0	6
0	1	0	$\frac{1}{15}$	$-\frac{4}{5}$	0	24
1	0	0	0	1	0	20
0	0	0	$\frac{17}{15}$	$\frac{32}{5}$	1	808

optimal

Basic	Non-basic
$x = 20$	$s_2 = 0$
$y = 24$	$s_3 = 0$
$s_1 = 6$	
$P = 808$	

The company should make 20 type A souvenirs, 24 type B souvenirs in order to get a maximum revenue of \$808. At the optimal point, there are 6 hours of labor time left, and no leftover capital. The maximum possible number of type A souvenirs, 20, is made.

(b) Solve the same problem using the Method of Corners and compare your answers.

Maximize $R = 20x + 17y$

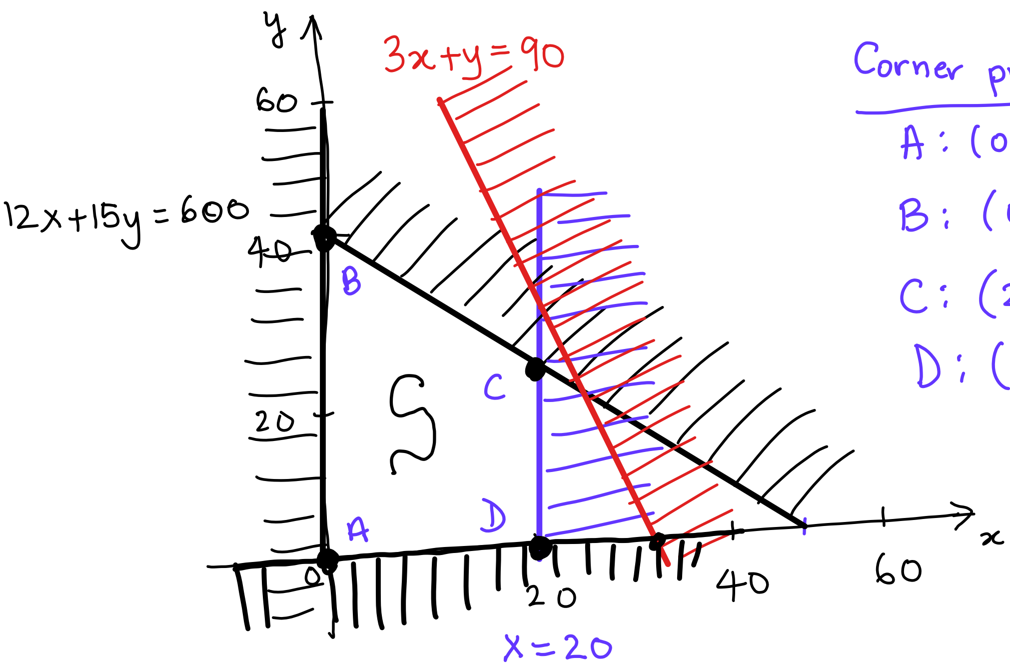
Subject to: $3x + y \leq 90$ (time)

$12x + 15y \leq 600$ (capital)

$x \leq 20$ (max of A)

$x \geq 0, y \geq 0$ (non-negative)

x = # of A
y = # of B
R = revenue generated



Corner pnt (x,y)	$R = 20x + 17y$
A: $(0,0)$	$R = 0$
B: $(0,40)$	$R = 680$
C: $(20,24)$	$R = 808$ * max
D: $(20,0)$	$R = 400$

Resource	Available	Used	Left-over
time	90 hrs	$3(20) + 24 = 84$	$90 - 84 = 6$ hrs
Capital	\$600	$12(20) + 15(24) = \$600$	$600 - 600 = \$0$

The company should make 20 type A souvenirs, 24 type B souvenirs in order to get a maximum revenue of \$808. At the optimal point, there are 6 hours of labor time left, and no leftover capital. The maximum possible number of type A souvenirs, 20, is made.

(c) The demand for souvenirs of type B increases during summer. If the company needs to produce at least twice as many B as A, what additional constraint would have been added to the set-up of the problem?

x = # of A
y = # of B
R = revenue generated

$y \geq 2x$

A	B
1	2^+
2	4^+
⋮	
x	$2x^+$

$\Rightarrow y \geq 2x$

13. Consider the sample space $S = \{p, q, r\}$.

(a) Determine the total number of events associated with this experiment.

total number of events $= 2^n$, n = number of outcomes $= 3$

$= 2^3 = \boxed{8 \text{ events}}$

(b) List all the events of this experiment.

$\{\ \}$, $\{p\}$, $\{q\}$, $\{r\}$, $\{p,q\}$, $\{p,r\}$, $\{q,r\}$, $\{p,q,r\}$

impossible event — simple events — certain event

(c) How many of these events are simple events? List one.

number of simple events = number of outcomes = 3

$\{p\}$, $\{q\}$, $\{r\}$ are the simple events

(d) Give an example of two events that are mutually exclusive. → if they have no common outcomes

$\{p,q\}$ and $\{r\}$ are mutually exclusive

14. Suppose there is an experiment with sample space $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e\}$ and events

$A =$ the event an even number is drawn. $A = \{2, 4, 6, 8\}$
$B = \{1, 3, 4, 5\}$
$C = \{1, 3, 5, 8\}$
$D =$ the event a letter from the phrase "fall break" is drawn. $D = \{a, b, e\}$

Answer each of the following.

(a) $(A \cap B) \cup C$ $A \cap B = \{4\}$ $C = \{1, 3, 5, 8\}$

$(A \cap B) \cup C = \{1, 3, 4, 5, 8\}$

(b) $A^C \cap B$ $A^C = \{0, 1, 3, 5, 7, 9, a, b, c, d, e\}$

$B = \{1, 3, 4, 5\}$

$A^C \cap B = \{1, 3, 5\}$

(c) $A \cap [(B \cup C)^C]$ $B \cup C = \{1, 3, 4, 5, 8\}$, $A = \{2, 4, 6, 8\}$

$(B \cup C)^C = \{0, 2, 6, 7, 9, a, b, c, d, e\}$

$A \cap [(B \cup C)^C] = \{2, 6\}$

(d) Determine the outcomes of event D

$D = \{a, b, e\}$

(e) Verbally describe the event A^C

$A^C =$ the event an odd number is drawn or a letter from "abcde" is selected.

