Math 308: Week-in-Review 13 (7.7 - 7.9)

7.7-7.9: Nonhomogeneous Linear Systems

Review

- For a nonhomogeneous system $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$, the general solution is $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$, where $\mathbf{x}_h$ is the general solution to the homogeneous system $\mathbf{x}' = A\mathbf{x}$, and $\mathbf{x}_p$ is a particular solution to the nonhomogeneous system.
- **Variation of Parameters:**
  - This method is general and can be used for any continuous $\mathbf{g}(t)$.
  - Assume $\mathbf{x}_p(t) = \mathbf{\Psi}(t)\mathbf{c}(t)$, where $\mathbf{\Psi}(t)$ is a fundamental matrix for the homogeneous system.
  - Then, $\mathbf{c}'(t) = \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)$, so $\mathbf{c}(t) = \int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt$.
  - Thus, $\mathbf{x}_p(t) = \mathbf{\Psi}(t)\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt$.
  - This method can be computationally intensive.
- **Undetermined Coefficients:**
  - This method is applicable when $\mathbf{g}(t)$ is a vector of polynomials, exponentials, sines, cosines, or linear combinations or products thereof.
  - Guess the form of $\mathbf{x}_p(t)$ based on the form of $\mathbf{g}(t)$:
    * For $\mathbf{g}(t) = \mathbf{a}e^{kt}$, guess $\mathbf{x}_p(t) = \mathbf{b}e^{kt}$.
    * For $\mathbf{g}(t) = \mathbf{a}t^m$, guess $\mathbf{x}_p(t) = \mathbf{b}_m t^m + \mathbf{b}_{m-1}t^{m-1} + \cdots + \mathbf{b}_0$.
    * For $\mathbf{g}(t) = \mathbf{a}\sin(\omega t) + \mathbf{c}\cos(\omega t)$, guess $\mathbf{x}_p(t) = \mathbf{b}\sin(\omega t) + \mathbf{d}\cos(\omega t)$.
  - If the guessed form is already part of the homogeneous solution, multiply the guess by t (or higher powers if necessary) to obtain a linearly independent form.
  - This method is often simpler than variation of parameters when applicable, but requires careful selection of the form of $\mathbf{x}_p(t)$.

1. Suppose

$$\mathbf{x}_1 = \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix}$$

are two indepedent solutions to the homogeneous equation $\mathbf{x}' = A\mathbf{x}$, and let

$$\Psi(t) = \begin{pmatrix} \varphi_1(t) & \psi_1(t) \\ \varphi_2(t) & \psi_2(t) \end{pmatrix}$$

be a fundamental matrix. Show that $\Psi'(t) = A\Psi(t)$. Show that the general solution of the homogeneous system can be written equivalently as

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \Psi(t)\mathbf{c}$$

where $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is an arbitrary constant vector.

$$\Psi'(t) = \begin{bmatrix} \varphi_1'(t) & \psi_1'(t) \\ \varphi_2'(t) & \psi_2'(t) \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}' & \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix}' \end{bmatrix}$$

$$= \begin{bmatrix} x_1' & x_2' \end{bmatrix}$$

$x_1' = Ax_1$

$x_2' = Ax_2$

$$= \begin{bmatrix} Ax_1 & Ax_2 \end{bmatrix}$$

$$= A\begin{bmatrix} x_1 & x_2 \end{bmatrix}$$

$$= A\begin{bmatrix} \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix} & \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} \end{bmatrix}$$

$$\boxed{\Psi'(t) = A\Psi(t)}$$

$$\Psi(t)\vec{C} = \begin{bmatrix} \varphi_1 & \psi_1 \\ \varphi_2 & \psi_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1\varphi_1 + c_2\psi_1 \\ c_1\varphi_2 + c_2\psi_2 \end{bmatrix} = c_1\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} + c_2\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

$$= c_1x_1 + c_2x_2$$

2. Consider the nonhomogeneous equation

$$\mathbf{x}' = \begin{bmatrix} -5 & 3 \\ 2 & -10 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}$$

Find the fundamental matrix and its inverse. Find a particular solution to the system and the general solution.

METHOD OF VARIATION OF PARAMETERS

$$x = x_h(t) + x_p(t) = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$$

where $x_1' = Ax_1$, $x_2' = Ax_2$ are independent solutions of $x' = Ax$.

$$x' = \begin{bmatrix} -5 & 3 \\ 2 & -10 \end{bmatrix} x$$

$$\lambda^2 + 15\lambda + 44 = 0$$

$$(\lambda+4)(\lambda+11) = 0$$

$\lambda_1 = -4$, $\lambda_2 = -11$: $A + 4I = \begin{bmatrix} -1 & 3 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad -a + 3b = 0$, $\quad a = 3b$

$$V_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$A + 11I = \begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 2a + b = 0$, $\quad 2a = -b$ $\quad V_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$x_1 = e^{-4t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad x_2 = e^{-11t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\Psi(t) = \begin{bmatrix} 3e^{-4t} & e^{-11t} \\ e^{-4t} & -2e^{-11t} \end{bmatrix} \Rightarrow \det \Psi(t) = -6e^{-15t} - e^{-15t} = -7e^{-15t}$$

$$\Psi^{-1}(t) = \frac{-1}{7} \begin{bmatrix} -2e^{4t} & -e^{4t} \\ -e^{11t} & 3e^{11t} \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 2e^{4t} & e^{4t} \\ e^{11t} & -3e^{11t} \end{bmatrix}$$

$$\Psi^{-1}(t) = \frac{1}{7}\begin{bmatrix} 2e^{4t} & e^{4t} \\ e^{11t} & -3e^{11t} \end{bmatrix}$$

$$X_p = \Psi(t)\int \Psi^{-1}(t)\,g(t)\,dt$$

$$\Psi^{-1}(t)\,g(t) = \frac{1}{7}\begin{bmatrix} 2e^{4t} & e^{4t} \\ e^{11t} & -3e^{11t} \end{bmatrix}\begin{bmatrix} e^{-t} \\ 0 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2e^{3t} \\ e^{10t} \end{bmatrix}$$

$$\int \Psi^{-1}(t)\,g(t)\,dt = \frac{1}{7}\int \begin{bmatrix} 2e^{3t} \\ e^{10t} \end{bmatrix} dt = \frac{1}{7}\begin{bmatrix} \frac{2}{3}e^{3t} \\ \frac{1}{10}e^{10t} \end{bmatrix}$$

$$X_p = \frac{1}{7}\begin{bmatrix} 3e^{-4t} & e^{-11t} \\ e^{-4t} & -2e^{-11t} \end{bmatrix}\begin{bmatrix} \frac{2}{3}e^{3t} \\ \frac{1}{10}e^{10t} \end{bmatrix} = \begin{bmatrix} \frac{3}{10}e^{-t} \\ \frac{1}{15}e^{-t} \end{bmatrix}$$

$$X(t) = X_h(t) + X_p(t)$$

$$= c_1\begin{bmatrix} 3e^{-4t} \\ e^{-4t} \end{bmatrix} + c_2\begin{bmatrix} e^{-11t} \\ -2e^{-11t} \end{bmatrix} + \begin{bmatrix} \frac{3}{10}e^{-t} \\ \frac{1}{15}e^{-t} \end{bmatrix}$$

3. Consider the non-homogeneous system

$$\mathbf{x}' = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} te^t \\ e^t \end{bmatrix}, \quad t > 0.$$

Verify that $\Psi(t) = \begin{bmatrix} e^t & \frac{t^2}{2}e^t \\ 0 & e^t \end{bmatrix}$ is a fundamental matrix. Then find the general solution of the system.

$$x' = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} te^t \\ e^t \end{bmatrix}, \quad t>0$$

$$\Psi(t) = \begin{bmatrix} e^t & \frac{t^2}{2}e^t \\ 0 & e^t \end{bmatrix} \Rightarrow \Psi'(t) = \begin{bmatrix} e^t & te^t + \frac{t^2}{2}e^t \\ 0 & e^t \end{bmatrix}$$

$$\Psi'(t) \overset{?}{=} A\Psi(t) : \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e^t & \frac{t^2}{2}e^t \\ 0 & e^t \end{bmatrix}$$

$$= \begin{bmatrix} e^t & \frac{t^2}{2}e^t + te^t \\ 0 & e^t \end{bmatrix}$$

$$x_1 = e^t\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_2 = e^t \begin{bmatrix} \frac{t^2}{2} \\ 1 \end{bmatrix}$$

$$\vec{x}_p = \Psi(t)\int \Psi^{-1}(t) g(t)\,dt \qquad g(t) = \begin{bmatrix} te^t \\ e^t \end{bmatrix}$$

$$\Psi^{-1}(t) = \frac{1}{e^{2t}}\begin{bmatrix} e^t & -\frac{t^2}{2}e^t \\ 0 & e^t \end{bmatrix}$$

$$\int \Psi^{-1}(t) g(t)\,dt = \int \begin{bmatrix} e^{-t}(te^t) - \frac{t^2}{2}e^{-t}(e^t) \\ e^{-t}(e^t) \end{bmatrix} dt$$

$$\vec{c}(t) = \int \begin{bmatrix} t - \frac{t^2}{2} \\ 1 \end{bmatrix} dt = \begin{bmatrix} \frac{t^2}{2} - \frac{t^3}{6} \\ t \end{bmatrix}$$

$$x_p(t) = \begin{bmatrix} e^t\left(\frac{t^2}{2} - \frac{t^3}{6}\right) + \frac{t^2}{2}e^t(t) \\ e^t(t) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{t^3}{3}e^t + \frac{t^2}{2}e^t \\ te^t \end{bmatrix} = te^t\begin{bmatrix} \frac{t^2}{3} + \frac{t}{2} \\ 1 \end{bmatrix}$$

4. Consider the system

$$x' = 3x + 2y + 3, \quad y' = 7x + 5y + 2t$$

(a) Find the fundamental matrix (b) Use undetermined coefficients to find a particular solution.

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 2t \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} t$$

$$\lambda_1 = 4+\sqrt{15}, \quad \lambda_2 = 4-\sqrt{15}$$

$$V_1 = \begin{bmatrix} 2 \\ 1+\sqrt{15} \end{bmatrix}, \quad V_2 = \begin{bmatrix} 2 \\ 1-\sqrt{15} \end{bmatrix}$$

$$\Psi(t) = \begin{bmatrix} 2e^{(4+\sqrt{15})t} & 2e^{(4-\sqrt{15})t} \\ (1+\sqrt{15})e^{(4+\sqrt{15})t} & (1-\sqrt{15})e^{(4-\sqrt{15})t} \end{bmatrix}$$

$$X_p = \vec{a}t + \vec{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$X_p' = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 3a_1 + 2a_2 \\ 7a_1 + 5a_2 \end{pmatrix} t + \begin{pmatrix} 3b_1 + 2b_2 \\ 7b_1 + 5b_2 \end{pmatrix} + \begin{pmatrix} 3 \\ 2t \end{pmatrix}$$

$$\boxed{\begin{aligned} * \; a_1 &= 3b_1 + 2b_2 + 3 \\ * \; a_2 &= 7b_1 + 5b_2 \end{aligned}}$$

$0 = 3a_1 + 2a_2$

$0 = 7a_1 + 5a_2 + 2$

$b_1 = 17$

$b_2 = -25$

$\rightarrow 3a_1 + 2a_2 = 0 \quad * 7$

$7a_1 + 5a_2 = -2 \quad * 3$

$21a_1 + 14a_2 = 0$

$21a_1 + 15a_2 = -6$

$a_2 = -6$

$3a_1 = -2a_2 = 12$

$a_1 = 4$

5. Use the Method of Undetermined Coefficients to find the general solution solution of the system

$$\mathbf{x}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 + e^t \\ -2 - 2e^t \end{bmatrix}$$

$$X_p = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} e^t$$

$$X(t) = c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + X_p$$

$$x' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x + \underbrace{\begin{bmatrix} 3 + e^t \\ -2 - 2e^t \end{bmatrix}}_{g(t)}$$

$$g(t) = \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^t$$

$$\lambda_{1,2} = \pm i \quad , \quad x_h = c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

$$\underbrace{A - iI}_{\lambda_1 = i} = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad a - ib = 0$$

$$a = ib$$

$$V_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$X(t) = c_1 \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(t) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(t) \right] + c_2 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(t) \right]$$

$$= c_1 \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

$$X_p(t) = \underbrace{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}_{X_p^{(1)}} + \underbrace{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^t}_{X_p^{(2)}}$$

$$X_p^{(1)\prime} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} X_p^{(1)} + \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} -a_2 \\ a_1 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$a_1 = 2, \ a_2 = 3$$

$$X_p^{(2)\prime} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} X_p^{(2)} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^t$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^t = \begin{bmatrix} -b_2 \\ b_1 \end{bmatrix} e^t + \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^t$$

$$b_1 = -b_2 + 1 \qquad b_2 = -b_2 + 1 - 2$$

$$b_2 = b_1 - 2 \nearrow \qquad 2b_2 = -1 \Rightarrow b_2 = -\frac{1}{2}$$

6. Use the Method of Undetermined Coefficients to determine the general solution of

$$\mathbf{x}' = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -3 \\ -1 \end{bmatrix} e^t$$

$\lambda_1 = 1,\ \lambda_2 = -1$

$V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ V_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$X_1 = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ X_2 = e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$X_h(t) = C_1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + C_2 \begin{bmatrix} 2e^{-t} \\ e^{-t} \end{bmatrix}$$

$$X_p = \underline{\vec{a}\, t e^t + \vec{b} e^t} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} t e^t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^t$$

$$X_p' = \vec{a} e^t + \vec{a} t e^t + \vec{b} e^t = \underbrace{(\vec{a} + \vec{b}) e^t + \vec{a} t e^t}$$

$$\parallel$$

$$\begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix} X_p + \begin{bmatrix} -3 \\ -1 \end{bmatrix} e^t$$

$$= \begin{pmatrix} -3a_1 + 4a_2 \\ -2a_1 + 3a_2 \end{pmatrix} t e^t + \begin{pmatrix} -3b_1 + 4b_2 \\ -2b_1 + 3b_2 \end{pmatrix} e^t + \begin{bmatrix} -3 \\ -1 \end{bmatrix} e^t$$

$* e^t *$

$a_1 + b_1 = -3b_1 + 4b_2 - 3$

$a_2 + b_2 = -2b_1 + 3b_2 - 1$

$* t e^t *$

$\left.\begin{array}{l} a_1 = -3a_1 + 4a_2 \\ a_2 = -2a_1 + 3a_2 \end{array}\right\}$ $4a_1 - 4a_2 = 0$

$2a_1 - 2a_2 = 0$

$a_1 = a_2$

$4b_1 + a_1 - 4b_2 = -3$

$2b_1 + a_1 - 2b_2 = -1 \quad * 2$

$4b_1 + 2a_1 - 4b_2 = -2$

$\downarrow\ a_1 = 1 \Rightarrow a_2 = 1$

$2b_1 - 2b_2 = -2$

$b_1 - b_2 = -1$

$b_1 = b_2 - 1 \Rightarrow \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow X_p = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t$

7. **Without solving for the coefficients**, determine the form of the particular solution $\mathbf{x}_p$ of the nonhomogeneous system

$$\mathbf{x}' = \underbrace{\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}}_{A} \mathbf{x} + \underbrace{\begin{bmatrix} -2\cos t + 4\sin t \\ 2\sin t \end{bmatrix}}_{g(t)}$$

$$x_p = \vec{a}\cos t + \vec{b}\sin t + \vec{c}\,t\cos t + \vec{d}\,t\sin t$$

$$\operatorname{tr} A = 0, \quad \det A = -4 + 5 = 1$$

$$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

$$x_h(t) = c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$