Week in Review
Math 152

Week 12

Test 3 Review

Common Exam III Prep.

2. Using the Alternating Series Estimation Theorem and the MacLaurin series for $f(x) = \sin x$, which of the following is an approximation to $\sin(1)$ so that the error is less than or equal to $\frac{1}{6!}$ with the fewest number of terms?

(a) $1 - \frac{1}{3!} + \frac{1}{5!}$ ← correct

(b) $1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!}$

(c) $1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} - \frac{1}{11!}$

(d) $1 - \frac{1}{2!} + \frac{1}{4!}$

(e) $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!}$

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

- $\sin 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$
- $|S - S_n| \leq |a_{n+1}|$

$= \frac{1}{(2n+3)!} \leq \frac{1}{6!}$

$(2n+3) \geq 6\ ;\ n \geq 1.5$

- $n = 2$

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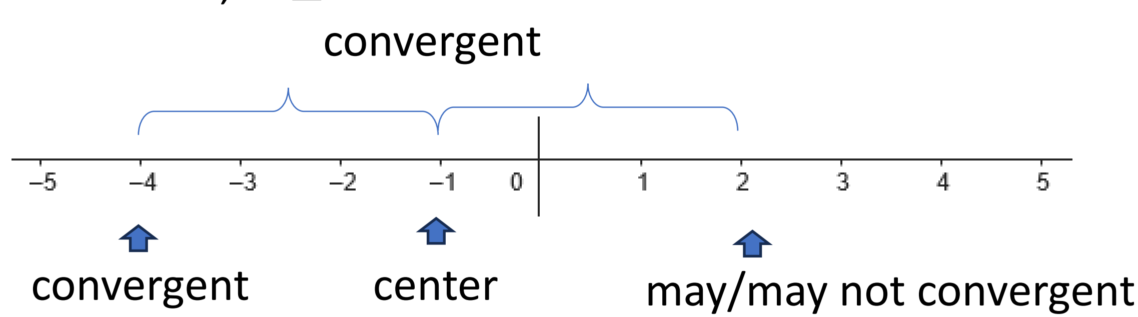
3. The series $\sum_{n=1}^{\infty} c_n(x+1)^n$ converges when $x = -4$. Which of the following series is guaranteed to converge?

(I) $\sum_{n=1}^{\infty} c_n \cdot 0^n$ (II) $\sum_{n=1}^{\infty} c_n$ (III) $\sum_{n=1}^{\infty} c_n 2^n$ (IV) $\sum_{n=1}^{\infty} c_n 3^n$

(a) I and II only
(b) I, II, and III only
(c) II and III only
(d) II, III, and IV only
(e) I, II, III and IV

If the radius of convergence is R,
the series is convergent for all x such that $|x+1| \leq R$
For $x = -4$, $|-4+1| \leq R$
Therefore, $R \geq 3$



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4. Which of the following statements is true regarding the series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$?

(a) The Ratio Test shows that the series is convergent.

(b) The Ratio Test shows that the series is divergent.

(c) The Limit Comparison Test shows that the series is convergent. ← correct

(d) The Limit Comparison Test shows that the series is divergent.

(e) The Limit Comparison Test is inconclusive.

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}}$$

Let $\frac{1}{n^2} = x$, then as $n \to \infty$, $x \to 0$.

- $\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\cos(x)}{1} == 1$

L'hospital's rule

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6. Which of the following statements is true for the following series?

(I) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$ (II) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^3}$ (III) $\sum_{n=1}^{\infty} \frac{e^n}{(-1)^n n}$

(a) I and III converge conditionally, and II diverges.

(b) I converges conditionally, II converges absolutely, and III diverges. ← correct

(c) I and II converge conditionally, and III diverges.

(d) I, II, and III converge conditionally.

(e) I, II, and III converge absolutely.

(1) $\frac{1}{n+1} \to 0$ (convergent)

(2) $\frac{1}{n(\ln n)^3} \to 0$ (convergent)

(3) $\frac{e^n}{n} \to \infty$ (divergent)

(1) $\frac{\lim_{n\to\infty} \frac{1}{n+1}}{\lim_{n\to\infty} \frac{1}{n}} = 1$ with $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ (divergent)

(2) $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx < \infty$(absolutely convergent)

(3) (divergent)

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1. Which of the following is true regarding the series $\sum_{n=1}^{\infty} \frac{5n \cdot 3^n}{4^n}$.

   (a) The Ratio Test limit is $\frac{3}{4}$, so the series converges.

   (b) The Ratio Test limit is $\frac{3}{4}$, so the series diverges.

   (c) The Ratio Test limit is $\frac{15}{4}$, so the series diverges.

   (d) The Ratio Test limit is $\frac{15}{4}$, so the series converges.

   (e) The Ratio Test limit is $\frac{9}{4}$, so the series diverges.

$\sum_{n=1}^{\infty} 5n\left(\frac{3}{4}\right)^n$ where $a_n = 5n\left(\frac{3}{4}\right)^n$

By ratio test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{5(n+1)\left(\frac{3}{4}\right)^{n+1}}{5n\left(\frac{3}{4}\right)^{n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)}{n}\left(\frac{3}{4}\right)\right| = \frac{3}{4} < 1$$

2. Find the Maclaurin series for the function $f(x) = x^2e^{-x^3}$.

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+2}}{n!}$

(b) $\sum_{n=0}^{\infty} \frac{x^{3n+6}}{n!}$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+6}}{n!}$

(d) $\sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$

(e) $\sum_{n=0}^{\infty} \frac{x^{3n+2}}{n!}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$x^2\left(e^{-x^3}\right) = x^2 \sum_{n=0}^{\infty} \frac{\left(-x^3\right)^n}{n!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+2}}{n!}$$

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3. Find the 3rd degree Taylor polynomial, $T_3(x)$, for the function $f(x) = \ln x$ centered at $a = 6$.

(a) $T_3(x) = \ln 6 + \frac{1}{6}(x-6) - \frac{1}{36}(x-6)^2 + \frac{1}{108}(x-6)^3$
(b) $T_3(x) = \ln 6 + \frac{1}{6}(x-6) - \frac{1}{72}(x-6)^2 + \frac{1}{648}(x-6)^3$
(c) $T_3(x) = \ln 6 + \frac{1}{6}(x-6) - \frac{1}{6}(x-6)^2 + \frac{1}{36}(x-6)^3$
(d) $T_3(x) = \ln 6 + \frac{1}{6}(x-6) - \frac{1}{36}(x-6)^2 + \frac{1}{216}(x-6)^3$
(e) $T_3(x) = \ln 6 + \frac{1}{6}(x-6) - \frac{1}{6}(x-6)^2 + \frac{1}{12}(x-6)^3$

Sol1

$$(\ln x)' = \frac{1}{x}$$
$$= \frac{1}{6-(6-x)} = \frac{\frac{1}{6}}{1-\frac{1}{6}(-(x-6))}$$
$$= \frac{1}{6}\left(1 + \frac{1}{6}(-(x-6)) + \left[\frac{1}{6}(-(x-6))\right]^2\right)$$
$$= \frac{1}{6}\left(1 - \frac{1}{6}(x-6) + \frac{1}{6^2}(x-6)^2\right)$$
$$\ln x = a_0 + \frac{1}{6}\left(x - \frac{1}{12}(x-6)^2 + \frac{1}{108}(x-6)^3\right)$$
$$\ln 6 = a_0$$
$$\ln x = \ln 6 + \frac{1}{6}x - \frac{1}{72}(x-6)^2 + \frac{1}{648}(x-6)^3$$

Sol 2.

$$f(6) + f'(6)(x-6) + \frac{f''(6)}{2}(x-6)^2 + \frac{f'''(6)}{6}(x-6)^3$$

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4. Find a power series representation for $f(x) = \dfrac{x}{x+4}$ and its radius of convergence.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{n+1}}{4^{n+1}}, \quad R = 4$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{n+1}}{4^{n+1}}, \quad R = \tfrac{1}{4}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{4^{n+1}}, \quad R = 4$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{4^{n+1}}, \quad R = \tfrac{1}{4}$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{4^{n+1}}, \quad R = 4$

$$f(x) = \frac{\left(\frac{x}{4}\right)}{1+\left(-\frac{x}{4}\right)}$$

$$= \frac{x}{4}\sum_{n=0}^{\infty}\left(-\frac{x}{4}\right)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{4}\right)^{n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x}{4}\right| < 1$$

$$|x| < 4 = R$$

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5. Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{n!(x+4)^n}{3^n}$.

(a) $\{0\}$

(b) $(-\infty, \infty)$

(c) $(-7, -1)$

(d) $(-4, 4)$

(e) $\{-4\}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(x+4)^{n+1}}{3^{n+1}}\cdot\frac{3^n}{n!(x+4)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)(x+4)}{3}\right| = \infty \text{ except for } x = -4$$

$RoC = 0$

$IoC = \{-4\}$

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6. Suppose that $0 < a_n < b_n$ for all $n \geq 1$. Which of the following statements is always true?

(a) If $\sum_{n=1}^{\infty} b_n$ is divergent, then so is $\sum_{n=1}^{\infty} a_n$.

(b) If $\sum_{n=1}^{\infty} a_n$ is convergent, then so is $\sum_{n=1}^{\infty} b_n$.

(c) If $\lim_{n\to\infty} b_n = 0$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(d) If $\sum_{n=1}^{\infty} a_n$ is divergent, then so is $\sum_{n=1}^{\infty} b_n$.

(e) If $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} b_n = 0$.

$0 < a_n < b_n$

$0 < \sum a_n \leq \sum b_n$

- $\sum a_n = \infty \Rightarrow \sum b_n = \infty$
- $\sum b_n < \infty \Rightarrow \sum a_n < \infty$

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7. Which of the following statments is true for the series $\sum_{n=1}^{\infty} \frac{3+\sin n}{n^5+1}$?

(a) The series converges since $\frac{3+\sin n}{n^5+1} < \frac{3}{n^5}$ and $\sum_{n=1}^{\infty} \frac{3}{n^5}$ converges.

(b) The series converges since $\frac{3+\sin n}{n^5+1} > \frac{2}{n^5}$ and $\sum_{n=1}^{\infty} \frac{2}{n^5}$ converges.

(c) The series diverges since $\frac{3+\sin n}{n^5+1} > \frac{2}{n^5}$ and $\sum_{n=1}^{\infty} \frac{2}{n^5}$ diverges.

(d) The series converges since $\frac{3+\sin n}{n^5+1} < \frac{4}{n^5}$ and $\sum_{n=1}^{\infty} \frac{4}{n^5}$ converges.

(e) None of these.

$$-1 \le \sin n \le 1$$
$$2 \le 3 + \sin n \le 4$$
$$\frac{2}{n^5+5} \le \frac{3+\sin n}{n^5+5} \le \frac{4}{n^5+5}$$
$$\sum \frac{3+\sin n}{n^5+5} \le \sum \frac{4}{n^5+5} \le \sum \frac{4}{n^5} < \infty$$

8. The alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+2)^2}$ converges.

Use the Alternating Series Estimation Theorem to determine an upper bound on the absolute value of the error in using s_5 to approximate the sum of the series.

(a) $\frac{1}{8}$

(b) $\frac{1}{9}$

(c) $\frac{1}{64}$

(d) $\frac{1}{81}$

(e) $\frac{1}{35}$

$$|S - S_5| \leq |a_{5+1}|$$
$$= \frac{1}{8^2}$$

9. Consider the series below, which statement is true regarding the absolute convergence of each series?

$$\text{(I)} \sum_{n=1}^{\infty} \frac{(-2)^n}{3^{n+1}} \qquad \text{(II)} \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+3}$$

(a) (I) converges but not absolutely, (II) converges absolutely.

(b) Both (I) and (II) converge but not absolutely.

(c) Both (I) and (II) converges absolutely.

(d) (I) converges abolutely, (II) diverges.

(e) (I) converges absolutely, (II) converges but not absolutely.

(I) $\sum \left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)^n$ =Geometric series w/ $r = -\frac{2}{3}$ (absolutely convergent)

(II) $\lim_{n\to\infty} \frac{n}{n^2+3} = 0$ (convergent alternating series)

$$\frac{\lim_{n\to\infty} \frac{n}{n^2+3}}{\lim_{n\to\infty} \frac{1}{n}} = 1 \text{ and } \sum \frac{1}{n} = \infty \qquad \Rightarrow \sum \frac{n}{n^2+3} = \infty$$

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10. For which series is the ratio test inconclusive?

(a) $\sum_{n=2}^{\infty} \frac{(-1)^n}{3^n\sqrt{\ln n}}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$

(c) $\sum_{n=1}^{\infty} ne^{-n}$

(d) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

(e) $\sum_{n=1}^{\infty} \frac{n+2}{n!}$

(a) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}\sqrt{\ln(n+1)}}{3^n\sqrt{\ln(n)}}\right| = 3 > 1$

(b) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1/(n+1)^5}{1/n^5}\right| = 1$ (inconclusive)

(c) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)}{e^{n+1}}\cdot\frac{e^n}{n}\right| = \frac{1}{e} < 1$

(d) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)}{2^{n+1}}\cdot\frac{2^n}{n}\right| = \frac{1}{2} < 1$

(e) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+3)}{(n+1)!}\cdot\frac{n!}{n+2}\right| = 0 < 1$

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11. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n}}{4^{2n}(2n)!}$

(a) $\cos(\frac{3}{4})$

(b) $3\cos(\frac{3}{4})$

(c) $\sin(\frac{3}{4})$

(d) $\cos 3$

(e) $\sin 3$

$$\sum_{n=0}^{\infty}(-1)^n \frac{\left(\frac{3}{4}\right)^{2n}}{(2n)!} = \cos\frac{3}{4}$$

12. The series $\sum_{n=2}^{\infty} c_n x^n$ converges when $x = 4$ and diverges when $x = -7$. What can be said about the convergence of the following series?

$$\text{(I)} \sum_{n=2}^{\infty} c_n 9^n \qquad \text{(II)} \sum_{n=2}^{\infty} c_n (-4)^n$$

(a) Both (I) and (II) are inconclusive.

(b) (I) diverges, (II) converges.

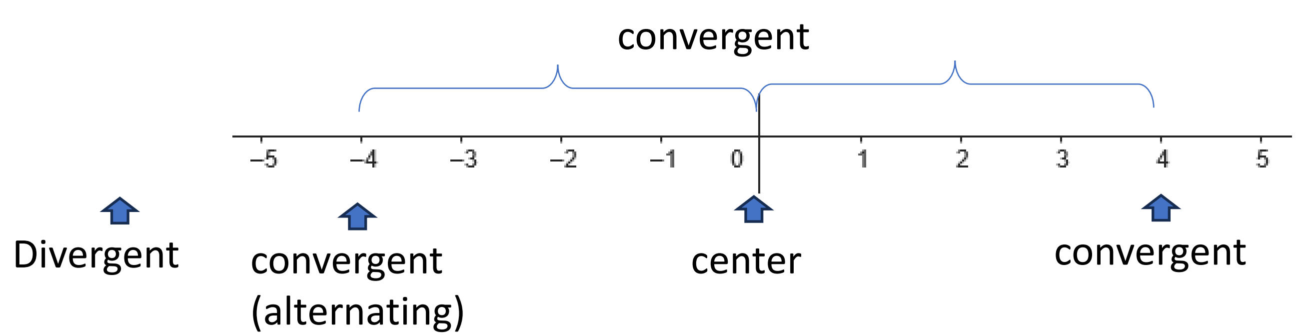
(c) (I) diverges, (II) is inconclusive.

(d) Both (I) and (II) converge.

(e) (I) is inconclusive, (II) converges.

(assume $c_n \geq 0$)

If the radius of convergence is R,
the series is convergent for all x such that $|x| \leq R$
For $x = 4, |4| \leq R$
For $x = -7, |-7| \geq R$
Therefore, $4 \leq R \leq 7$



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13. Find the coefficient of x^3 in the Maclaurin series for the function $f(x) = \sin(2x)$.

(a) $\frac{4}{3}$

(b) $-\frac{4}{3}$

(c) $\frac{2}{3}$

(d) $-\frac{2}{3}$

(e) $\frac{1}{3}$

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin 2x = \sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n+1}}{(2n+1)!}$$

$$n = 1 \Rightarrow (-1)^1 \frac{(2x)^3}{3!} = -\frac{8}{6}x^3$$

14. Which of the following statements is true for the series $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+1}$?

(a) The series converges absolutely.

(b) The series converges but not absolutely.

(c) The series diverges by the alternating series test.

(d) The series diverges by the test for divergence.

(e) None of these.

$\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$ (divergent by divergence test)

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15. Evaluate the indefinite integral $\int \arctan(4x^3)\, dx$ as a Maclaurin series.

(a) $C + \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+2} x^{6n+6}}{(2n+1)(2n+2)}$

(b) $C + \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1} x^{6n+6}}{(2n+1)(2n+2)}$

(c) $C + \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+2} x^{6n+4}}{(2n+1)(6n+4)}$

(d) $C + \sum_{n=0}^{\infty} \frac{(-1)^n 4^{6n+4} x^{6n+4}}{(2n+1)(6n+4)}$

(e) $C + \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1} x^{6n+4}}{(2n+1)(6n+4)}$

$$(\operatorname{atan} x)' = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$
$$= \sum_{n=0}^{\infty} (-x^2)^n$$
$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$
$$\operatorname{atan} x = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$
$$\operatorname{atan}(0) = C = 0$$

$$\operatorname{atan}(4x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (4x^3)^{2n+1}}{2n+1}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1} x^{6n+3}}{2n+1}$$

$$\int \operatorname{atan}(4x^3)\, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1} x^{6n+4}}{(2n+1)(6n+4)}$$

16. (12 pts) Find (a) the Radius of convergence and

(b) Interval of convergence of the power series $\sum_{n=2}^{\infty} \frac{(3x-1)^n}{8^n(n-1)}$.

RoC:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3x-1)^{n+1}}{8^{n+1}n}\cdot\frac{8^n(n-1)}{(3x-1)}\right| = \lim_{n\to\infty}\left|\frac{3x-1}{8}\right| < 1$$

$$3\left|x-\frac{1}{3}\right| < 8$$

$$\left|x-\frac{1}{3}\right| < \frac{8}{3} \;\Rightarrow R = \frac{8}{3} \text{ with } \left(-\frac{7}{3}, 3\right)$$

IoC

$@3x - 1 = 8 \; (x = 3)$

$\sum_{n=2}^{\infty}\frac{1}{n-1} = \infty$ (Harmonic series)

$@3x - 1 = -8 \; (x = -\frac{7}{3})$

$\sum_{n=2}^{\infty}\frac{(-1)^n}{n-1} < \infty$ (Alternating series w/ $a_n \to 0$)

IoC : $\left[-\frac{8}{3}+\frac{1}{3}, \frac{8}{3}+\frac{1}{3}\right) = \left[-\frac{7}{3}, 3\right)$

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17. (8 pts) Find the Taylor Series for $f(x) = \dfrac{1}{x^3}$ centered at $x = 2$.

$\int f dx = C_1 - \frac{1}{2x^2}$

$\iint f dx dx = C_2 + C_1 x + \frac{1}{2x}$

$= C_2 + C_1 x + \frac{1/2}{2-(2-x)}$ $\qquad = C_2 + C_1 + \frac{1/4}{1-\frac{2-x}{2}}$

$= C_1 + C_2 x + \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{(-1)(x-2)}{2}\right)^n$

$= C_1 + C_2 x + \sum_{n=0}^{\infty}\frac{(-1)^n (x-2)^n}{2^{n+2}}$

$\int f dx = C_1 + \sum_{n=1}^{\infty}\frac{(-1)^n n (x-2)^{n-1}}{2^{n+2}}$

$f = \sum_{n=2}^{\infty}\frac{(-1)^n n(n-1)(x-2)^{n-2}}{2^{n+2}}$

Let $k = n - 2 \Rightarrow n = k + 2$

$f = \sum_{k=0}^{\infty}\frac{(-1)^k (k+2)(k+1)(x-2)^k}{2^{k+4}}$

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18. (12 pts) Express $\int_0^{1/2} \cos(x^2)\, dx$ as an infinite series.

$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

$\cos x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}$

$\int_0^{1/2} \cos x^2\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int_0^{1/2} x^{4n} dx$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left[\frac{x^{4n+1}}{4n+1}\right]_0^{1/2}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{1}{(4n+1)2^{4n+1}}$

Common Exam III Prep.

19. (8 pts) Determine whether the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}+5}{n^3-2n}$ converges or diverges. Support your answer.

By limit comparison,

$$\frac{\lim_{n\to\infty} \frac{\sqrt{n}+5}{n^3-2n}}{\lim_{n\to\infty} \frac{\sqrt{n}}{n^3}} = 1 \text{ with } \sum_{n=0}^{\infty} \frac{\sqrt{n}}{n^3} < \infty \text{ (p-series)}$$

Therefore $\sum_{n=0}^{\infty} \frac{\sqrt{n}+5}{n^3-2n} < \infty$