Math 308: Week-in-Review 9 (6.6, 5.1, 5.2)

1 6.6: Convolution

Review

- The convolution of two functions $f(t)$ and $g(t)$, denoted $(f * g)(t)$, is defined by

$$(f * g)(t) = \int_0^t f(x)g(t-x)\,dx$$

for $t \geq 0$, assuming both functions are zero for $t < 0$.

- The Laplace transform of a convolution $(f * g)(t)$ is

$$\mathcal{L}\{(f * g)(t)\} = F(s) \cdot G(s),$$

where $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$.

- This property simplifies solving differential equations by converting convolution in the time domain to multiplication in the s-domain.

- Convolution is commutative: $f * g = g * f$ so

$$(f * g)(t) = \int_0^t f(x)g(t-x)\,dx = \int_0^t f(t-x)g(x)\,dx.$$

1. Find the following convolutions using the definition only
   (a) $e^{2t} * e^{4t}$
   (b) $t^2 * t$,

SOLUTION 1

(a) Using the convolution definition:

$$(e^{2t} * e^{4t})(t) = \int_0^t e^{2x} e^{4(t-x)}\, dx$$

Simplify the integrand:

$$e^{2x}e^{4(t-x)} = e^{2x}e^{4t-4x} = e^{4t}e^{2x-4x} = e^{4t}e^{-2x}$$

So the integral becomes:

$$e^{4t}\int_0^t e^{-2x}\, dx$$

Evaluate the integral:

$$\int e^{-2x}\, dx = -\frac{1}{2}e^{-2x}$$

$$\left[-\frac{1}{2}e^{-2x}\right]_0^t = -\frac{1}{2}e^{-2t} - \left(-\frac{1}{2}\right) = -\frac{1}{2}e^{-2t} + \frac{1}{2}$$

Thus,

$$e^{4t}\left(\frac{1}{2} - \frac{1}{2}e^{-2t}\right) = \frac{1}{2}e^{4t} - \frac{1}{2}e^{4t-2t} = \frac{1}{2}e^{4t} - \frac{1}{2}e^{2t}$$

Final answer:

$$\boxed{e^{2t} * e^{4t} = \frac{e^{4t} - e^{2t}}{2}}$$

(b) Using the definition:

$$(t^2 * t)(t) = \int_0^t x^2(t-x)\, dx$$

Expand the integrand:

$$x^2(t-x) = tx^2 - x^3$$

So,

$$\int_0^t (tx^2 - x^3)\, dx = t\int_0^t x^2\, dx - \int_0^t x^3\, dx = t\left[\frac{x^3}{3}\right]_0^t - \left[\frac{x^4}{4}\right]_0^t = t\cdot\frac{t^3}{3} - \frac{t^4}{4} = \frac{t^4}{3} - \frac{t^4}{4} = \frac{t^4}{12}$$

Final answer:

$$\boxed{t^2 * t = \frac{t^4}{12}}$$

2. Using the Laplace transform (instead of the definition) compute the following convolutions

   (a) $u_2(t) * u_3(t)$

   (b) $t^2 * t$,

SOLUTION 2

(a) Laplace transforms:

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}$$

$$\mathcal{L}\{u_2 * u_3\} = \frac{e^{-2s}}{s} \cdot \frac{e^{-3s}}{s} = \frac{e^{-5s}}{s^2}$$

Inverse Laplace transform:

$$\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s^2}\right\} = u_5(t)(t-5)$$

Final answer:

$$\boxed{u_2(t) * u_3(t) = u_5(t)(t-5)}$$

(b) Laplace transforms:

$$\mathcal{L}\{t^2\} = \frac{2}{s^3}, \quad \mathcal{L}\{t\} = \frac{1}{s^2}$$

$$\mathcal{L}\{t^2 * t\} = \frac{2}{s^3} \cdot \frac{1}{s^2} = \frac{2}{s^5}$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^5}\right\} = 2 \cdot \frac{t^4}{24} = \frac{t^4}{12}$$

Final answer:

$$\boxed{t^2 * t = \frac{t^4}{12}}$$

3. In each of the following cases find a function (or generalized function) $g(t)$ that satisfies the equality for $t \geq 0$

   (a) $t^2 * g(t) = t^5$

   (b) $1 * 1 * g(t) = t^3$

   (c) $1 * g(t) = t$

SOLUTION 3

(a) Take Laplace transforms:

$$\mathcal{L}\{t^2\} \cdot \mathcal{L}\{g(t)\} = \mathcal{L}\{t^5\}$$

$$\frac{2}{s^3}G(s) = \frac{120}{s^6} \implies G(s) = \frac{120}{s^6} \cdot \frac{s^3}{2} = \frac{60}{s^3}$$

$$g(t) = \mathcal{L}^{-1}\left\{\frac{60}{s^3}\right\} = 30t^2$$

Final answer:

$$\boxed{g(t) = 30t^2}$$

(b) First, $1 * 1 = t$, so $t * g(t) = t^3$. Laplace transforms:

$$\mathcal{L}\{t\} \cdot \mathcal{L}\{g(t)\} = \mathcal{L}\{t^3\}$$

$$\frac{1}{s^2}G(s) = \frac{6}{s^4} \implies G(s) = \frac{6}{s^2}$$

$$g(t) = \mathcal{L}^{-1}\left\{\frac{6}{s^2}\right\} = 6t$$

Final answer:

$$\boxed{g(t) = 6t}$$

(c) Laplace transforms:

$$\mathcal{L}\{1\} \cdot \mathcal{L}\{g(t)\} = \mathcal{L}\{t\}$$

$$\frac{1}{s}G(s) = \frac{1}{s^2} \implies G(s) = \frac{1}{s}$$

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

Final answer:

$$\boxed{g(t) = 1}$$

4. Write the inverse Laplace transform in terms of a convolution integral

$$F(s) = \frac{s^2}{(s+2)^3(s+5)^2}$$

SOLUTION 4 *Split $F(s)$:*

$$F(s) = \frac{1}{(s+2)^3} \cdot \frac{s^2}{(s+5)^2}$$

Define inverse transforms:

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^3}\right\} = \frac{1}{2}t^2 e^{-2t}$$

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\left\{\frac{s^2}{(s+5)^2}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{(s+5)^2 - 10(s+5) + 25}{(s+5)^2}\right\} \\ &= \mathcal{L}^{-1}\left\{1 - \frac{10}{s+5} + \frac{25}{(s+5)^2}\right\} \\ &= \delta(t) - 10e^{-5t} + 25te^{-5t} \end{aligned}$$

Apply convolution theorem accounting for Dirac delta: (note that $w(t) * \delta(t) = w(t)$ for any function $w(t)$)

$$f(t) = g(t) + (g * k)(t)$$

where $k(t) = -10e^{-5t} + 25te^{-5t}$. For $t > 0$:

$$f(t) = \frac{1}{2}t^2 e^{-2t} + \int_0^t \frac{1}{2}x^2 e^{-2x}\left(-10e^{-5(t-x)} + 25(t-x)e^{-5(t-x)}\right) dx$$

Final answer:

$$\boxed{\frac{1}{2}t^2 e^{-2t} + \int_0^t \frac{1}{2}x^2 e^{-2x}\left(-10e^{-5(t-x)} + 25(t-x)e^{-5(t-x)}\right) dx}$$

5. Solve the initial value problem

$$y'' - 3y' + 2y = h(t), \;\; y(0) = 2, \;\; y'(0) = -1.$$

SOLUTION 5 *Apply the Laplace transform to the ODE:*

$$(s^2Y(s) - sy(0) - y'(0)) - 3(sY(s) - y(0)) + 2Y(s) = H(s)$$

Substitute $y(0) = 2$, $y'(0) = -1$*:*

$$(s^2 - 3s + 2)Y(s) - 2s + 7 = H(s)$$

Solve for $Y(s)$*:*

$$Y(s) = \frac{H(s)}{(s-1)(s-2)} + \frac{2s-7}{(s-1)(s-2)}$$

Partial fractions:

$$\frac{2s-7}{(s-1)(s-2)} = \frac{5}{s-1} - \frac{3}{s-2}$$

For the non-homogeneous term:

$$\frac{1}{(s-1)(s-2)} = \frac{-1}{s-1} + \frac{1}{s-2}$$

Inverse Laplace:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-2)}\right\} = -e^t + e^{2t}$$

By the convolution theorem:

$$\mathcal{L}^{-1}\left\{\frac{H(s)}{(s-1)(s-2)}\right\} = h(t) * (-e^t + e^{2t}) = \int_0^t \left(-e^{t-x} + e^{2(t-x)}\right) h(x)\,dx$$

Combine all terms:

$$y(t) = 5e^t - 3e^{2t} + \int_0^t \left(e^{2(t-x)} - e^{t-x}\right) h(x)\,dx$$

Final Answer:

$$\boxed{5e^t - 3e^{2t} + \int_0^t \left(e^{2(t-x)} - e^{t-x}\right) h(x)\,dx}$$

2 5.1–5.2: Power Series Solutions of Linear Differential Equations

Review

- A power series solution of a linear differential equation assumes the solution can be written as

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where x_0 is the center of the series (often $x_0 = 0$), and a_n are coefficients to be determined.

- The derivatives of the power series are:

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2},$$

obtained by term-by-term differentiation, assuming the series converges in some interval.

- For a linear differential equation of the form $P(x)y'' + Q(x)y' + R(x)y = 0$, substitute the power series for y, y', and y'' into the equation, equate coefficients of like powers of $(x - x_0)$, and solve for the recurrence relation among the a_n.

- A point x_0 is an *ordinary point* if $P(x_0) \neq 0$ and the coefficients $Q(x)/P(x)$ and $R(x)/P(x)$ are analytic at x_0. In this case, the series solution converges in some interval around x_0.

- The radius of convergence of the series solution is at least as large as the distance from x_0 to the nearest singular point (where $P(x) = 0$), determined by analyzing the coefficient functions.

- Solutions typically yield two linearly independent series $y_1(x)$ and $y_2(x)$, whose Wronskian $W[y_1, y_2](x_0) \neq 0$ confirms their independence.

6. Determine the radius of convergence for the power series

(a) $\sum_{n=0}^{\infty} \dfrac{x^{3n}}{n!}$

(b) $\sum_{n=1}^{\infty} \dfrac{(-1)^n n^3 (x-1)^n}{4^n}$

SOLUTION 6

(a) Ratio test:

$$\lim_{n\to\infty} \frac{|x|^{3(n+1)}/(n+1)!}{|x|^{3n}/n!} = |x|^3 \cdot \frac{1}{n+1} \to 0 < 1 \quad \forall x$$

Final answer:

$$R = \infty$$

(b) Ratio test:

$$\lim_{n\to\infty} \frac{(n+1)^3|x-1|^{n+1}/4^{n+1}}{n^3|x-1|^n/4^n} = \frac{|x-1|}{4} \cdot \left(\frac{n+1}{n}\right)^3 \to \frac{|x-1|}{4} < 1$$

Final answer:

$$R = 4$$

7. For the equation $y'' - xy' + xy = 0$

    (a) Determine a lower bound for the radius of convergence about $x_0 = 0$.

    (b) Seek its power series solution about $x_0 = 0$. Find the recurrence relation.

    (c) Find the general term of each solution $y_1(x)$ and $y_2(x)$.

    (d) Find the first four terms in each of the solutions. Show that $W[y_1, y_2](0) \neq 0$.

SOLUTION 7

(a) Coefficients are analytic, no singular points. Final answer:

$$R = \infty \quad \textit{(valid for all real numbers)}$$

(b) Assume $y = \sum_{n=0}^{\infty} a_n x^n$*:*

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - n a_n + a_{n-1}]x^n = 0$$

$$a_2 = 0, \quad a_{n+2} = \frac{n a_n - a_{n-1}}{(n+2)(n+1)}$$

Final answer:

$$a_{n+2} = \frac{n a_n - a_{n-1}}{(n+2)(n+1)}, \quad n \geq 1.$$

(c) Assume $y(0) = 1$, $y'(0) = 0$ *i.e.* ($a_0 = 1$, $a_1 = 0$)*:*

$$a_2 = 0, \quad a_3 = -\frac{1}{6}, \quad a_4 = 0, \quad a_5 = -\frac{1}{40}, \quad a_6 = \frac{1}{180}$$

$$y_1(x) = 1 - \frac{1}{6}x^3 - \frac{1}{40}x^5 + \frac{1}{180}x^6 + \cdots$$

Assume now that $y(0) = 0$, $y'(0) = 1$ *i.e.* ($a_0 = 0$, $a_1 = 1$)*:*

$$a_3 = \frac{1}{6}, \quad a_4 = -\frac{1}{12}, \quad a_5 = \frac{1}{40}$$

$$y_2(x) = x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{40}x^5 + \cdots$$

Final answer:

$$y_1(x) = 1 - \frac{1}{6}x^3 - \frac{1}{40}x^5 + \frac{1}{180}x^6 + \cdots, \quad y_2(x) = x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{40}x^5 + \cdots$$

(d) First four terms:

$$y_1(x) = 1 - \frac{1}{6}x^3 - \frac{1}{40}x^5 + \frac{1}{180}x^6 + \cdots, \quad y_2(x) = x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{40}x^5 + \cdots$$

Wronskian at $x = 0$:

$$W(0) = y_1(0)y_2'(0) - y_1'(0)y_2(0) = 1 - 0 \cdot 0 = 1 \neq 0$$

so $y_1(x)$ and $y_2(x)$ are linearly independent.