



# Week in Review

## Math 152

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### **Week 02**

The Substitution Rule  
Area Between Curves



# Integration formulas

$$\int 0 dx =$$

$$\int dx =$$

$$\int x^r dx = \quad (r \neq -1)$$

$$\int \cos x dx =$$

$$\int \sin x dx =$$

$$\int \sec^2 x dx =$$

$$\int \sec x \tan x dx =$$

$$\int \csc x \cot x dx =$$

$$\int e^x dx =$$

$$\int b^x dx = (0 < b, b \neq 1)$$

$$\int \frac{1}{x} dx =$$

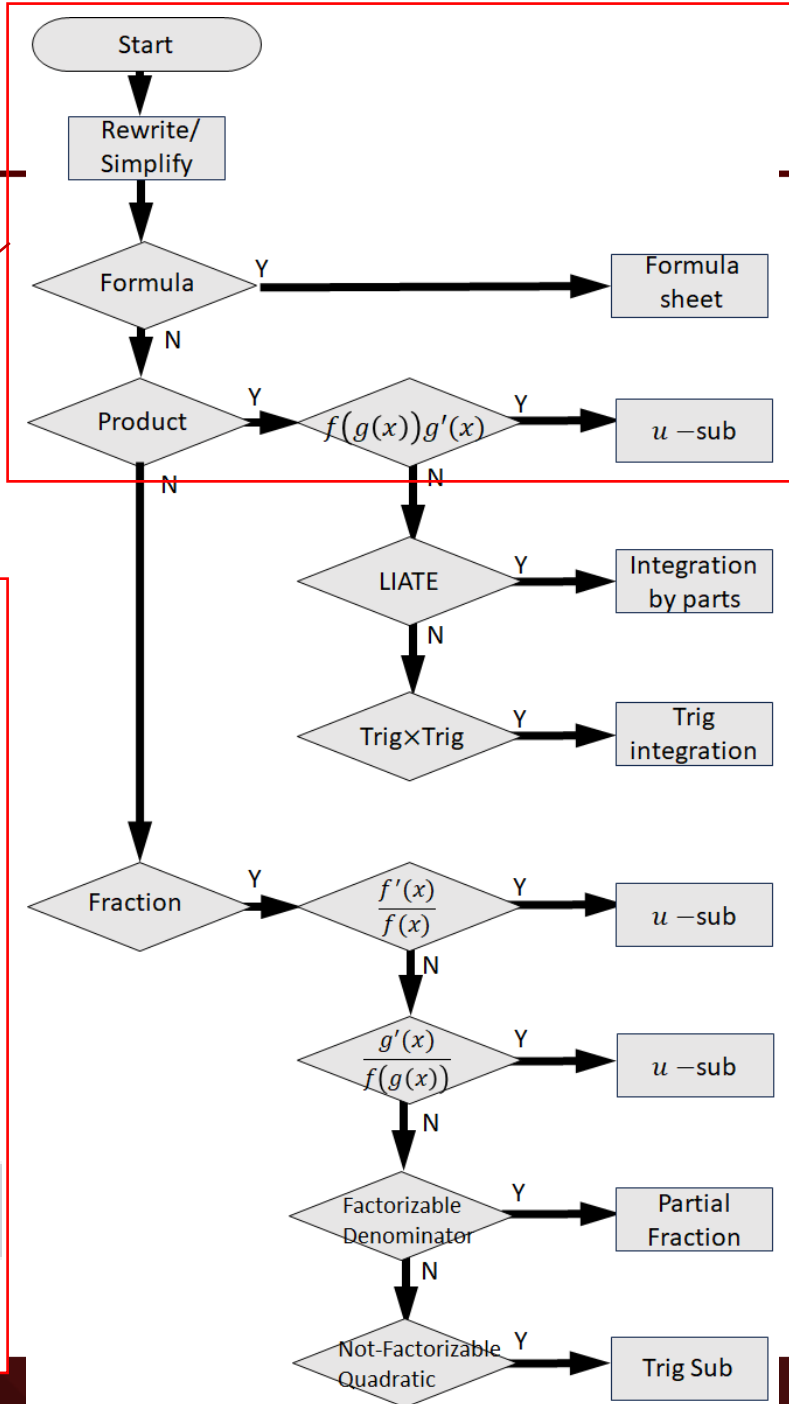
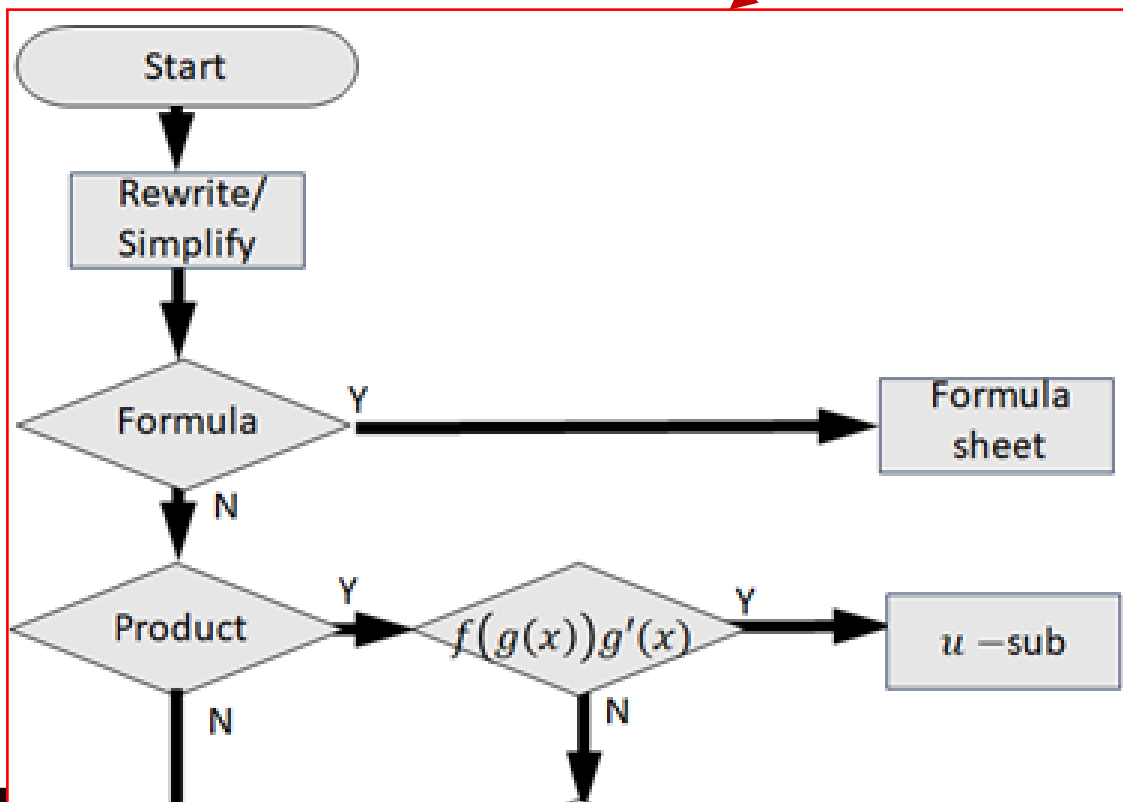
$$\int \frac{1}{1+x^2} dx =$$

$$\int \frac{1}{\sqrt{1-x^2}} dx =$$

$$\int \frac{1}{x\sqrt{x^2-1}} dx =$$



# Integration Workflow





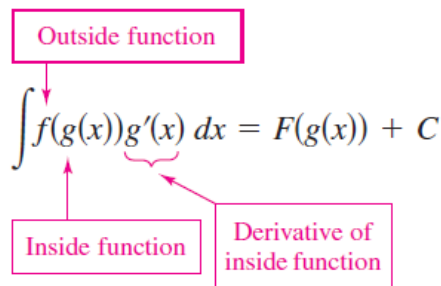
# u –substitution to find an antiderivative

## Method of u-substitution (Euler)

□ Differentiation  $\xleftrightarrow{\text{inverse}}$  Integration

- In differentiation : Chain Rule :  $\frac{d}{dx} [F(g(x))] = F'(g(x))g'(x)$
- In integration:  $\int F'(g(x))g'(x)dx = F(g(x)) + C$

### How to apply u –sub



1. Pattern recognition

- ***The integrand is a product of two functions (composite fun \* inner fun derivative)***

- $f$  = outer function (usually more complex)
- $g(x)$  = inner function (usually simpler)

2. Substitute the inner function as  $u$

- $u = g(x)$  then  $du = g'(x)dx$
- Include the constant term

3. Rewrite the integral in  $u$ :  $\int f(u)du$

4. Integrate in  $u$ :  $\int f(u)du = F(u) + C$

5. Substitute back to  $x$ :  $\int f(g(x))g'(x)dx = F(g(x)) + C$

$$\int x^3 \sqrt{2 + x^4} dx$$

$$f = \sqrt{2 + x^4}$$

$$g = 2 + x^4$$

$$u = 2 + x^4$$

$$du = 4x^3 dx$$

$$x^3 dx = 1/4 du$$

$$\int f(u)du = \int \sqrt{2 + x^4} x^3 dx = \frac{1}{4} \int \sqrt{u} du$$

$$F(u) = \frac{1}{4} \left( \frac{2}{3} u^{3/2} + C \right)$$

$$F(x) = \frac{1}{6} (2 + x^4)^{3/2} + C$$



# Problem

Compute  $\int x^3 \sqrt{2 + x^4} dx$

(a) None of these

(b)  $\frac{1}{4} (2 + x^4)^{3/2} + C$

(c)  $\frac{1}{6} (2 + x^4)^{3/2} + C$

(d)  $\frac{8}{3} (2 + x^4)^{3/2} + C$

(e)  $\frac{3}{8} (2 + x^4)^{3/2} + C$

Rewrite :

Id  $f$  and  $g'$

- $f =$
- $g =$
- $g' =$

u-sub

Solve



# Problem

Evaluate  $\int x^3 \sqrt{x^2 + 1} dx$ .

(a)  $\frac{1}{5}(x^2 + 1)^{\frac{5}{2}} - \frac{1}{3}(x^2 + 1)^{\frac{3}{2}} + C$

(b)  $\frac{1}{3}(x^2 + 1)^{\frac{3}{2}} + C$

(c)  $3x^2 \sqrt{x^2 + 1} + \frac{x^4}{\sqrt{x^2 + 1}} + C$

(d)  $\frac{2}{5}(x^2 + 1)^2 - \frac{2}{3}(x^2 + 1) + C$

(e) None of these

Rewrite :

Id  $f$  and  $g'$

- $f =$
- $g =$
- $g' =$

u-sub

Evaluate  
the integral:



# Problem

$$\int \frac{\sin x}{(1 + \cos x)^3} dx =$$

(a)  $\frac{1}{2(1 + \cos x)^2} + C$

(b)  $\frac{1}{(1 + \cos x)^2} + C$

(c)  $\frac{-1}{2(1 + \cos x)^2} + C$

(d)  $\frac{1}{4(1 + \cos x)^4} + C$

(e)  $\frac{-1}{4(1 + \cos x)^4} + C$

Rewrite :

Id  $f$  and  $g'$

- $f =$
- $g =$
- $g' =$

u-sub

Evaluate  
the integral:



# u – substitution for a definite integral

## Change of Variables for Definite Integrals

### ❑ Back substitution (Not recommended)

If  $\int_a^b f(g(x))g'(x)dx = F(g(x)) + C$  then

$$\int_a^b f(g(x))g'(x)dx = [F(g(x))]_a^b = F(g(a)) - F(g(b)) = [F(u)]_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(u)du$$

### ❑ Complete substitution $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u) du$

- Substitute  $u = g(x)$  then  $du = g'(x)dx$
- **Change the limits with respect to u:**  $\int_a^b \rightarrow \int_{g(a)}^{g(b)}$  ( $g$  should be one to one)
- Rewrite the integral in  $u$ :  $\int_{g(a)}^{g(b)} f(u) du$
- Integrate in  $u$ :  $\int_{g(a)}^{g(b)} f(u) du = [F(u)]_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$

**Example:** Evaluate  $\int_1^e \frac{\ln x}{x} dx$  by back substitution.

Find an antiderivative: Rewrite :  $\int \ln x \left(\frac{1}{x} dx\right)$

u-sub  $u = \ln x ; du = \frac{1}{x} dx$

Antiderivative  $\int u du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$

Evaluate the integral:  $\int_1^e \frac{\ln x}{x} dx = \left[\frac{1}{2}(\ln x)^2\right]_1^e = \frac{1}{2}(\ln e)^2 - \frac{1}{2}(\ln 1)^2 = \frac{1}{2}(1) - \frac{1}{2}(0) = \frac{1}{2}$





## $u$ –substitution for a definite integral

Evaluate  $\int_1^e \frac{\ln x}{x} dx$  by complete substitution.

Rewrite :

Id  $f(g)$  and  $g'$

- $f(g) =$
- $g =$
- $g' =$

u-sub

complete substitution  
for the limits

Evaluate the integral:



# Problem

Compute  $\int_0^{\sqrt{\pi}} x \sin(\pi - x^2) dx$  Rewrite :

(a)  $-\frac{\sin \sqrt{\pi}}{2}$

(b)  $-2$

(c)  $-1$

(d)  $1$

(e)  $2$

Id  $f$  and  $g'$

- $f =$

- $g =$

- $g' =$

u-sub

complete substitution  
for the limits

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

Evaluate the integral:



# Problem

Evaluate  $\int_0^{\pi/4} \frac{\sec^2(\theta)}{2 + \tan(\theta)} d\theta$  Rewrite :

(a)  $\ln\left(\frac{4}{3}\right)$

(b)  $\ln\left(\frac{\pi}{4}\right)$

(c)  $\ln\left(\frac{\pi}{8}\right)$

(d)  $\ln\left(\frac{\pi}{12}\right)$

(e)  $\ln\left(\frac{3}{2}\right)$

Id  $f$  and  $g'$

- $f =$
- $g =$
- $g' =$

u-sub

complete substitution  
for the limits

Evaluate the integral:



# Problem

If  $f$  is continuous and  $\int_0^{16} f(x) dx = 8$ ,

find  $\int_0^4 xf(x^2) dx$ .

- (a) 16
- (b) 2
- (c) 8
- (d) 64
- (e) 4

Rewrite :

Id  $f$  and  $g'$

- 
- $g =$
- $g' =$

u-sub

complete substitution  
for the limits

Evaluate the integral:



# Problem

Evaluate  $\int_{\sqrt{2}}^2 \frac{4x}{x^2 - 1} dx$

- (a)  $2 \ln 3$
- (b)  $2 \ln 2 - 2 \ln \sqrt{2}$
- (c)  $\ln 3 - 1$
- (d)  $2 \ln 3 - 2$
- (e)  $2 \ln 2$

Rewrite :

Id  $f$  and  $g'$

- 
- $g =$
- $g' =$

u-sub

complete substitution  
for the limits

Evaluate the integral:



# Problem

Evaluate  $\int_{\pi^2/16}^{\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}} dx$

(a)  $\frac{\sqrt{2}}{2} + 1$

(b)  $2 + \sqrt{2}$

(c)  $\frac{\sqrt{2}}{2} - 1$

(d)  $-2 + \sqrt{2}$

(e)  $1 - \sqrt{2}$

Rewrite :

Id  $f$  and  $g'$

- 
- $g =$
- $g' =$

u-sub

complete substitution  
for the limits

Evaluate the integral:



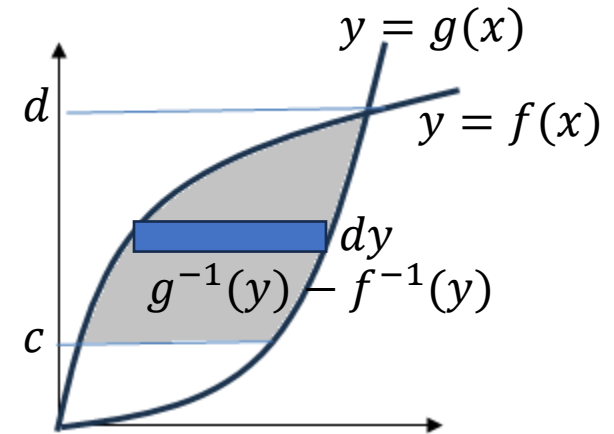
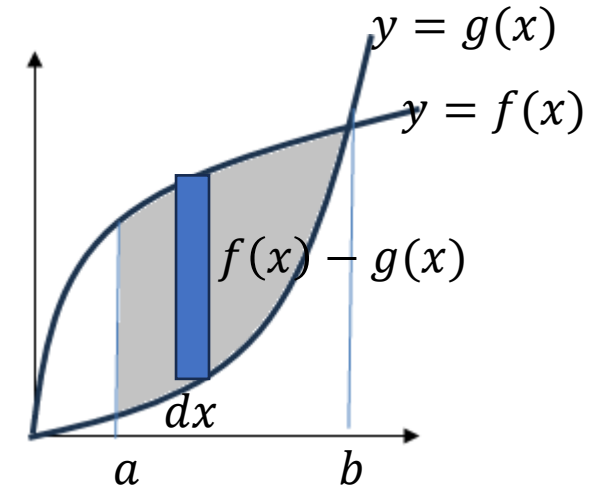
# Area between curves

The integration domain is  $x$  axis  $\Rightarrow dx$

1. Plot the graphs
2. Draw an infinitesimal strip
3. Find the area of the infinitesimal strip
  - [Function of  $x$ ] $dx$
  - [Large fun – Small fun] $dx$
  - $[f(x) - g(x)]dx$
4. Find the upper/lower limits
  - Solve the equations if needed
  - In this case,  $[a, b]$  is given
5. Integrate the area of infinitesimal strips
  - $\int_a^b [f(x) - g(x)]dx$

The integration domain is  $y$  axis  $\Rightarrow dy$

2. the area of the infinitesimal strip
  - [Function of  $y$ ] $dy$
  - [Large fun,  $g$  – Small fun,  $f$ ] $dy$
  - $[g^{-1}(y) - f^{-1}(y)]dy$
  - $\int_c^d [g^{-1}(y) - f^{-1}(y)]dy$





# Area between curves $f(x)$ and $g(x)$ , $f > g$ (Example)

Find the area between the curves:  $y = x^2$  and  $y = x + 6$ .

When  $f(x) \geq g(x)$

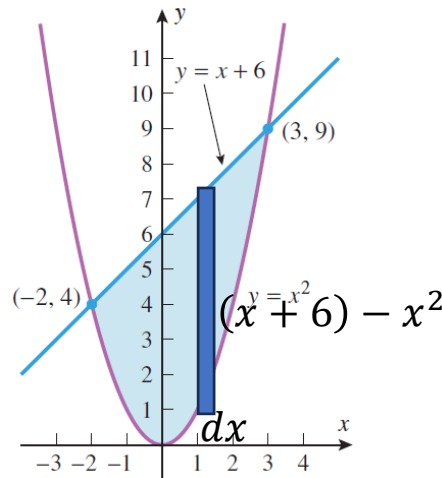
1. Plot the graphs

2. Draw an infinitesimal strip

3. Find the area of the infinitesimal strip

4. Find the upper/lower limits

5. Integrate the area of infinitesimal strips



$$[(x + 6) - x^2]dx$$

(Need limits for x)

$$x^2 = x + 6$$

$$(x + 2)(x - 3) = 0$$

$$\Rightarrow x = -2, 3$$

$$= \int_{-2}^3 (x + 6 - x^2) dx$$

$$= \left[ \frac{1}{2}x^2 - 6x - \frac{1}{3}x^3 \right]_{-2}^3$$

$$= \frac{1}{2}[x^2]_{-2}^3 - 6[x]_{-2}^3 - \frac{1}{3}[x^3]_{-2}^3$$

$$= \frac{9-4}{2} - 6(3 - (-2)) - \frac{27-(-8)}{3}$$

$$= -\frac{235}{6}$$

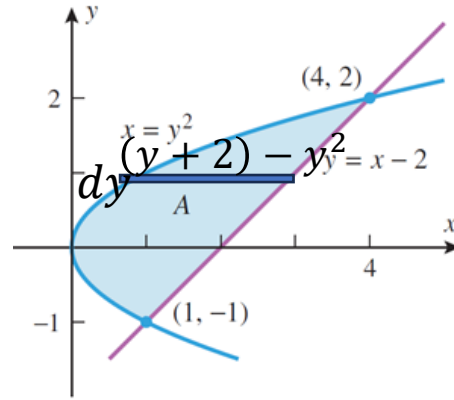




# Area between curves $f(y)$ and $g(y)$ , $f > g$ (Example)

When  $f(y) \geq g(y)$

1. Plot the graphs



2. Draw an infinitesimal strip

3. Find the area of the infinitesimal strip

4. Find the upper/lower limits

5. Integrate the area of infinitesimal strips

Area between  
 $x = y^2$  and  $y = x - 2$

$$[y + 2 - y^2]dy$$

(Need limits for  $y$ )

- $y + 2 = y^2; y^2 - y - 2 = 0$   
 $(y + 1)(y - 2) = 0$   
 $\Rightarrow y = -1, 2$

$$\begin{aligned} & \int_{-1}^2 (y + 2 - y^2) dy \\ &= \left[ \frac{1}{2}y^2 + 2y - \frac{1}{3}y^3 \right]_{-1}^2 \\ &= \frac{1}{2}(2^2 - 1) + 2(2 + 1) - \frac{1}{3}(2^3 + 1) \\ &= \frac{3}{2} + 6 - \frac{9}{3} = \frac{3}{2} + 3 = \frac{9}{2} \end{aligned}$$



## Problem

Find the area bounded by

$$y = e^x, y = e^{-x}, x = 0, \text{ and } x = 1.$$

(a)  $e + \frac{1}{e} - 2$

(b)  $e - \frac{1}{e}$

(c)  $e + \frac{1}{e} + 2$

(d)  $1 + \frac{1}{e}$

(e)  $1 + \frac{1}{e} - 2$



## Problem

Which of the following integrals gives the area of the region bounded by the curves  $x = y^2$  and  $x = 6 - y$ ?

(a)  $\int_{-3}^2 (6 - y - y^2) dy$

(b)  $\int_{-3}^2 (y^2 - 6 + y) dy$

(c)  $\int_4^9 (6 - x - \sqrt{x}) dx$

(d)  $\int_4^9 (\sqrt{x} - 6 + x) dx$

(e)  $\int_4^9 (6 - y - y^2) dy$



## Problem

Find the area bounded by  $y + x^2 = 6$  and  $y + 2x - 3 = 0$ .

(a)  $\frac{16}{3}$

(b)  $\frac{40}{3}$

(c)  $\frac{20}{3}$

(d)  $\frac{32}{3}$

(e) None of the above



## Problem

Find the area of the region bounded by  $x = y^2$  and  $x = y + 2$ .

(a)  $\frac{9}{2}$

(b)  $\frac{3}{2}$

(c)  $\frac{19}{6}$

(d)  $\frac{16}{3}$

(e) None of the above