1. Find the Cartesian equation of the curve.

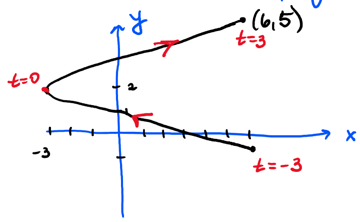
(a) $x = t^2 - 3$, $y = t + 2$, $-3 \le t \le 3$.

Let's eliminate t from the equation for y: $y = t+2$, $t = y-2$

Now, I'll plug the equation for t into $x(t) = t^2 - 3$ or

$x = (y-2)^2 - 3$ ← parabola

vertex @ $(-3, 2)$.



$t=-3$: $x(-3) = 9-3 = 6$, $y(-3) = -3+2 = -1$

$t=0$: $x(0) = -3$, $y(0) = 2$.

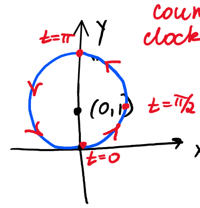
$t=3$: $x(3) = 9-3 = 6$, $y(3) = 3+2 = 5$.

(b) $x = \sin t$, $y = 1 - \cos t$, $0 \le t \le 2\pi$.

Trig identity $\sin^2 t + \cos^2 t = 1$.

$x = \sin t$, $y = 1 - \cos t \longrightarrow \cos t = 1 - y$

$\underbrace{x^2}_{\sin^2 t} + \underbrace{(1-y)^2}_{\cos^2 t} = 1$ ← circle of radius 1 centered @ $(0,1)$



counter clock wise direction.

$t=0$: $x(0) = \sin 0 = 0$, $y(0) = 1 - \cos 0 = 0$ → $(0,0)$

$t=\frac{\pi}{2}$: $x(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$, $y(\frac{\pi}{2}) = 1 - \cos\frac{\pi}{2} = 1$ → $(1,1)$

$t=\pi$: $x(\pi) = \sin \pi = 0$, $y(\pi) = 1 - \cos \pi = 1 - (-1) = 2$ → $(0,2)$

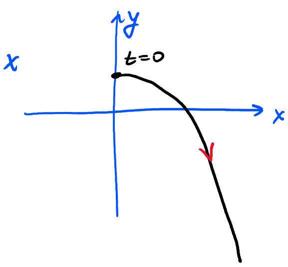
(c) $x = \sqrt{t}$, $y = 1 - t$.

$x \geq 0$

$t = 1 - y$ plug into the equation for x

$x = \sqrt{1-y}$, $x \geq 0$.

parabola, vertex @ $(0,1)$

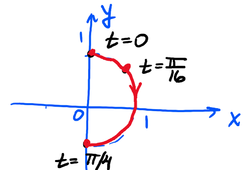


2. Sketch the curve given by $x = \sin(4t)$, $y = \cos(4t)$, $0 \le t \le \pi/4$ and indicate the direction of the curve that is traced as the parameter increases.

$x = \sin 4t$, $y = \cos 4t$, $0 \le t \le \frac{\pi}{4}$

trig identity $\cos^2 4t + \sin^2 4t = 1$

$x^2 + y^2 = 1$ – circle, radius 1
center @ $(0,0)$



clock wise direction.

$t = 0$: $x(0) = \sin 0 = 0$, $y(0) = \cos 0 = 1$ $(0,1)$

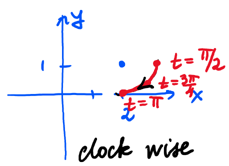
$t = \frac{\pi}{16}$: $x\left(\frac{\pi}{16}\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $y\left(\frac{\pi}{16}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\rightarrow \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$t = \frac{\pi}{4}$: $x\left(\frac{\pi}{4}\right) = \sin\frac{4\pi}{4} = \sin\pi = 0$, $y\left(\frac{\pi}{4}\right) = \cos\frac{4\pi}{4} = \cos\pi = -1$ $\rightarrow (0,-1)$

3. Describe the motion of the particle with position (x, y) given as $x = 2 + \sin t$, $y = 1 + \cos t$ as t varies from $\pi/2$ to π.

$x = 2 + \sin t$
$y = 1 + \cos t$, $\frac{\pi}{2} \le t \le \pi$

$\sin t = x - 2$
$\cos t = y - 1$ $\longrightarrow$ $\sin^2 t + \cos^2 t = 1 \rightarrow (x-2)^2 + (y-1)^2 = 1$
circle, radius 1
center @ $(2,1)$



clock wise direction.

$\frac{\pi}{2} \le t \le \pi$

$t = \frac{\pi}{2} \rightarrow$ $x = 2 + \sin\frac{\pi}{2} = 2 + 1 = 3$, $y = 1 + \cos\frac{\pi}{2} = 1$ $(3,1)$

$t = \pi \rightarrow$ $x = 2 + \sin\pi = 2$, $y = 1 + \cos\pi = 1 + (-1) = 0$ $(2,0)$

$t = \frac{3\pi}{4} \rightarrow$ $x = 2 + \sin\frac{3\pi}{4} = 2 + \frac{\sqrt{2}}{2}$, $y = 1 + \cos\frac{3\pi}{4} = 1 - \frac{\sqrt{2}}{2}$.

$$L = \int_a^b |\vec{r}\,'(t)|\,dt, \quad |\vec{r}\,'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}, \quad a \le t \le b.$$

4. Set up, but do not evaluate, the integral for the length of the curve $y = t + e^{-t}$, $y = t^2 + t$, $\underbrace{1 \le t \le 2}$.

$$\vec{r}(t) = \langle t + e^{-t}, t^2 + t \rangle$$
$$\vec{r}\,'(t) = \langle 1 - e^{-t}, 2t + 1 \rangle$$
$$|\vec{r}\,'(t)| = \sqrt{(1 - e^{-t})^2 + (2t+1)^2}$$
$$= \sqrt{1 - 2e^{-t} + e^{-2t} + 4t^2 + 4t + 1}$$
$$L = \int_1^2 \sqrt{1 - 2e^{-t} + e^{-2t} + 4t^2 + 4t + 1}\,dt$$

5. The curve C is given by $x = 3t - t^3$, $y = 3t^2$, $\underbrace{0 \le t \le 2}$.

(a) Find the exact length of the curve.

$$L = \int_0^2 \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$$
$$x = 3t - t^3, \quad x'(t) = 3 - 3t^2 = 3(1 - t^2)$$
$$y = 3t^2, \quad y'(t) = 6t$$
$$\sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{9(1-t^2)^2 + 36t^2} = 3\sqrt{(1-t^2)^2 + 4t^2}$$
$$= 3\sqrt{1 - 2t^2 + t^4 + 4t^2} = 3\sqrt{1 + 2t^2 + t^4} = 3\sqrt{(1+t^2)^2} = 3(1+t^2)$$

$$L = \int_0^2 3(1+t^2)\,dt = 3\left(t + \frac{t^3}{3}\right)_0^2 = 6 + 8 = \boxed{14}$$

$x = 3t - t^3$, $y = 3t^2$, $\sqrt{[x'(t)]^2 + [y'(t)]^2} = 3(1+t^2)$, $0 \le t \le 2$

(b) Find the area of the surface obtained by rotating the curve C about the x-axis.

$$S.A. = 2\pi \int_a^b y(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$S.A. = 2\pi \int_0^2 (3t^2)\, 3(1+t^2)\, dt = 18\pi \int_0^2 t^2(1+t^2)\, dt$$

$$= 18\pi \int_0^2 (t^2 + t^4)\, dt = 18\pi \left(\frac{t^3}{3} + \frac{t^5}{5} \right)_0^2$$

$$= 18\pi \left(\frac{8}{3} + \frac{32}{5} \right) = 18\pi\, \frac{40+96}{15} = \frac{18\pi \cdot 136}{15} = \frac{6\pi(136)}{5} = \boxed{\frac{816\pi}{5}}$$

(c) Find the area of the surface obtained by rotating the curve C about the y-axis.

$$S.A. = 2\pi \int_a^b x(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

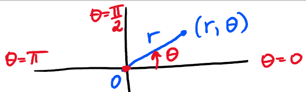
$$= 2\pi \int_0^2 (3t - t^3)3(1+t^2)\, dt = 6\pi \int_0^2 (3t + 3t^3 - t^3 - t^5)\, dt$$

$$= 6\pi \int_0^2 (3t + 2t^3 - t^5)\, dt$$

$$= 6\pi \left(\frac{3t^2}{2} + \frac{2t^4}{4} - \frac{t^6}{6} \right)_0^2$$

$$= 3\pi \left(3t^2 + t^4 - \frac{t^6}{3} \right)_0^2 = 3\pi \left(3 \cdot 4 + 16 - \frac{64}{3} \right)$$

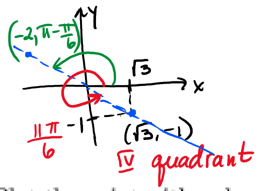
$$= 84\pi - 64\pi = \boxed{20\pi}$$



$x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2=r^2$, $\tan\theta = \frac{y}{x}$

6. Give the polar coordinates for the Cartesian point $(\sqrt{3}, -1)$. Find polar coordinates (r, θ) of the point when $r > 0$ and when $r < 0$. $\theta = \frac{3\pi}{2}$

Cartesian $(\sqrt{3}, -1)$



polar coordinates

$r^2 = 3+1 = 4$

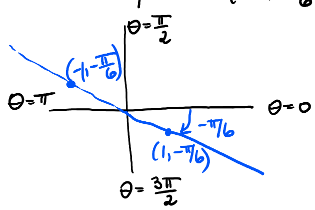
$\tan\theta = -\frac{1}{\sqrt{3}} \longrightarrow \theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

$r > 0 \longrightarrow \boxed{\left(2, \frac{11\pi}{6}\right)}$

$r < 0 \longrightarrow \left(-2, \pi - \frac{\pi}{6}\right) = \boxed{\left(-2, \frac{5\pi}{6}\right)}$

7. Plot the point with polar coordinates $(-1, -\pi/6)$. Find Cartesian coordinates of the point.

polar $\left(-1, -\frac{\pi}{6}\right)$



cartesian

$x = r\cos\theta = (-1)\cos\left(-\frac{\pi}{6}\right) = -1 \cdot \cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$

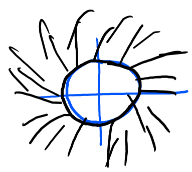
$y = r\sin\theta = (-1)\sin\left(-\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$

$\boxed{\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)}$

8. Sketch the region given by

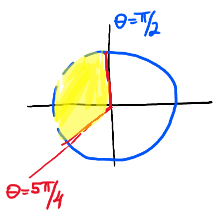
(a) $r \geq 2$

r=2 – circle of radius 2, centered @ (0,0)



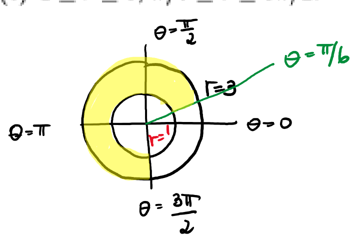
the region outside the circle + the circle.

(b) $0 \leq r < 3$, $\pi/2 \leq \theta \leq 5\pi/4$



(the circle is not included)

(c) $1 \leq r \leq 3$, $\pi/6 \leq \theta \leq 3\pi/2$.



the circles are included.

9. Find a Cartesian equation of the curve.

(a) $r^2 = 5$

$r^2 = x^2 + y^2 \longrightarrow \boxed{x^2+y^2=5}$

(b) $r = 4\sec\theta$

$\frac{r}{\sec\theta} = 4$ or $r\cos\theta = 4 \longrightarrow \boxed{x=4}$

(c) $r(r) = 4r\cos\theta$

$\underbrace{r^2}_{x^2+y^2} = 4\underbrace{r\cos\theta}_{x} \longrightarrow x^2+y^2 = 4x \longrightarrow \boxed{(x-2)^2+y^2=4}$

(d) $r^2\sin(2\theta) = 1$

trig identity $\sin 2\theta = 2\sin\theta\cos\theta$

$r^2\sin 2\theta = 2r^2\sin\theta\cos\theta = 2(\underbrace{r\cos\theta}_{x})(\underbrace{r\sin\theta}_{y})$

$2xy = 1 \rightarrow \boxed{xy = \frac{1}{2}}$

$x = r\cos\theta$

$y = r\sin\theta$

10. Find a polar equation for the curve

(a) $y = x$

$$\frac{r\sin\theta}{r\cos\theta} = \frac{r\cos\theta}{r\cos\theta} \longrightarrow \frac{\sin\theta}{\cos\theta} = 1 \quad \text{or} \quad \tan\theta = 1$$

$$\boxed{\theta = \frac{\pi}{4}}$$

(b) $x^2 + y^2 = 4y$

$x^2 + y^2 = r^2$

$y = r\sin\theta$

$$\Rightarrow r^2 = 4r\sin\theta$$

$$\boxed{r = 4\sin\theta}$$

(c) $4y^2 = x$

$y = r\sin\theta$

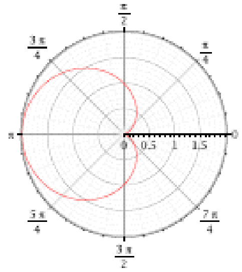
$x = r\cos\theta$

$$\longrightarrow \frac{4r^2\sin^2\theta}{r} = \frac{r\cos\theta}{r} \longrightarrow 4r\sin^2\theta = \cos\theta$$

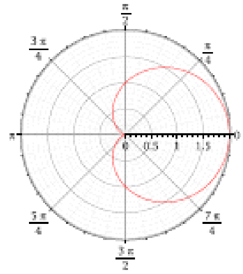
$$\boxed{r = \frac{\cos\theta}{4\sin^2\theta}}$$

Curves in Polar Coordinates

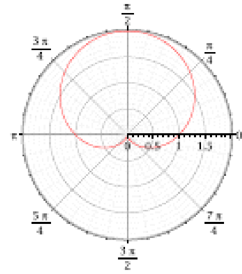
$r = 1 - \cos(\theta)$



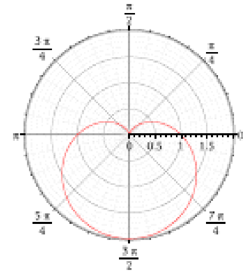
$r = 1 + \cos(\theta)$



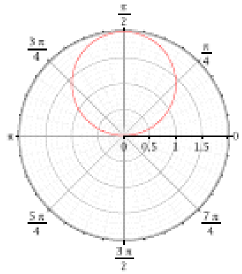
$r = 1 + \sin(\theta)$



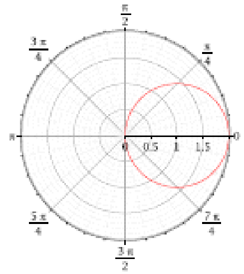
$r = 1 - \sin(\theta)$



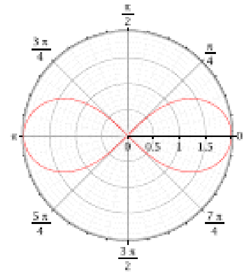
$r = 2\sin(\theta)$



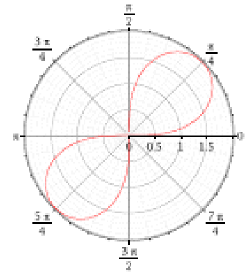
$r = 2\cos(\theta)$



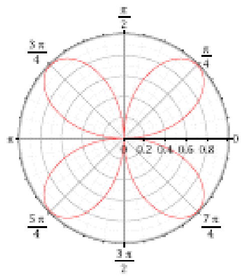
$r^2 = \cos(2\theta)$



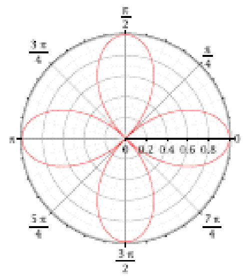
$r^2 = \sin(2\theta)$



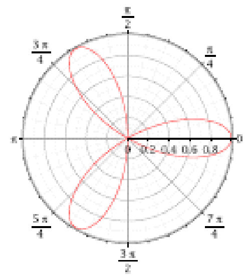
$r = \sin(2\theta)$



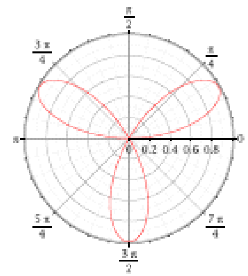
$r = \cos(2\theta)$



$r = \cos(3\theta)$

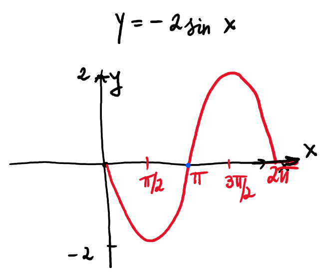


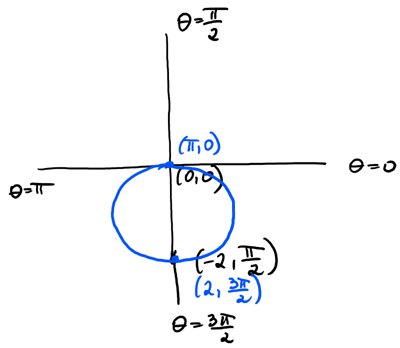
$r = \sin(3\theta)$



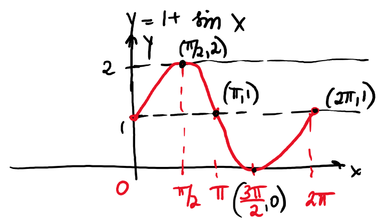
11. Sketch the curve with the given polar equation.

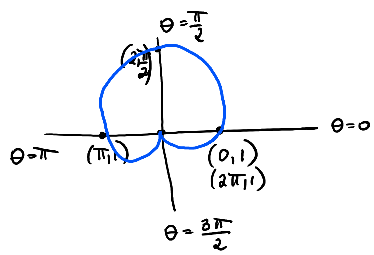
(a) $r = -2\sin\theta$





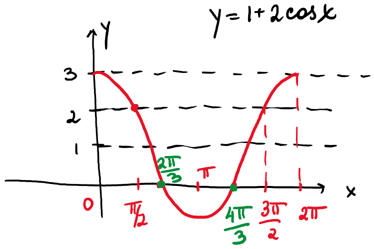
(b) $r = 1 + \sin\theta$





(c) $r = 1 + 2\cos\theta$

$y = 1 + 2\cos x$



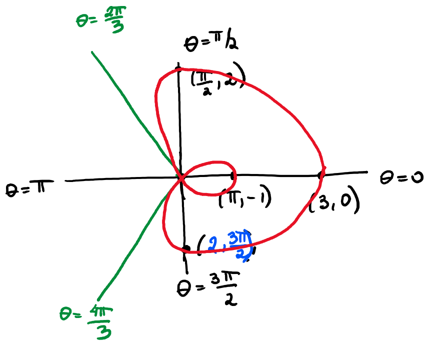
$1 + 2\cos x = 0$

$\cos x = -\frac{1}{2}$

$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$

points
$(0, 3)$
$(\frac{\pi}{2}, 2)$
$(\frac{2\pi}{3}, 0)$
$(\pi, -1)$
$(\frac{4\pi}{3}, 0)$
$(\frac{3\pi}{2}, 2)$
$(2\pi, 3)$



(d) $r = 3\sin(3\theta)$

$y = 3\sin 3x$

zeroes: $\sin 3x = 0$

$3x = \pi n, \quad n = 0, 1, 2, 3, \ldots$

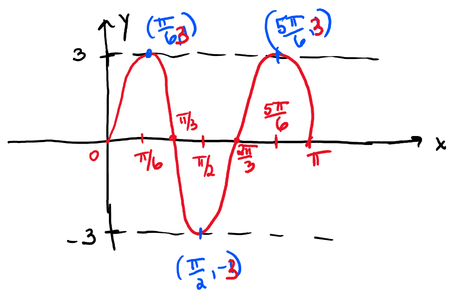
$x = \frac{\pi n}{3} \longrightarrow 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$

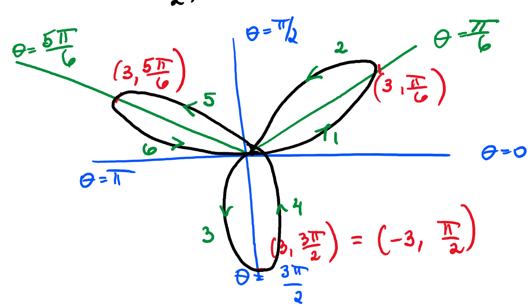
$\sin 3x = 1 \longrightarrow 3x = \frac{\pi}{2} + 2\pi n \longrightarrow x = \frac{\pi}{6} + \frac{2\pi n}{3}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots$

$\sin 3x = -1 \longrightarrow 3x = \frac{3\pi}{2} + 2\pi n \longrightarrow x = \frac{\pi}{2} + \frac{2\pi n}{3}$

$x = \frac{\pi}{2}, \pi, \ldots$





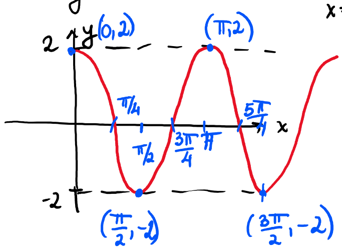
(e) $r = 2\cos(2\theta)$

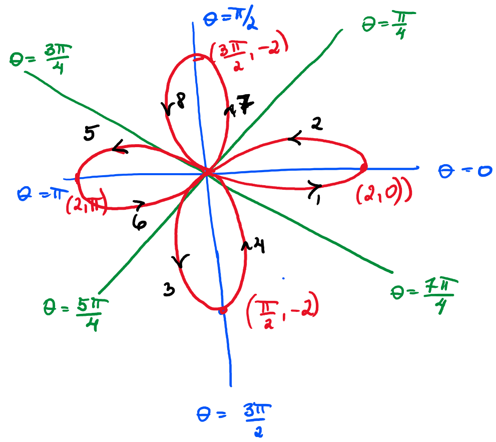
$\cos 2x = 0$

$2x = \frac{\pi}{2} + \pi n$

$x = \frac{\pi}{4} + \frac{\pi n}{2} \longrightarrow x = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4}$

$y = 2\cos 2x$





(f) $r^2 = 9\sin(2\theta)$

$y^2 = 9\sin 2x$